DEPARTMENT OF MATHEMATICS
UNIVERSITY OF OXFORD

HISTORY OF MATHEMATICS

LECTURE NOTES

Submitted by:
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BO1 History of Mathematics
Lecture I
Introduction

MT 2022 Week 1

Contact details

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Summary

- The nature of history
- How can we organise/break down the history of mathematics?
- Rough overview of the course
- Arrangements: lectures, classes, the nature of the course
- Some advice on reading and taking notes

What is history (of mathematics)?

History, particularly the history of mathematics, is often written in a very ‘narrative’ way, as a sequence of events with dates, detailing the achievements of major figures, telling a linear story of how we got to where we are today

Although this approach can be useful as a first approximation to the real story, there are several things wrong with it ...

... history is much more than a catalogue of events

What is history (of mathematics)?

When we study history, we may start by addressing the what, the when, and the who, but we are also interested in the how, and, perhaps most importantly, the why

The major figures in the history of mathematics are not the only people to have contributed to mathematics, so we might need to expand our notion of what counts as 'mathematics'

The story of mathematics is not linear: there are false starts and dead-ends, twists and turns, parallel developments; it is not a story of relentless progress: there are fallow periods and mistakes — but these too have shaped mathematics in their own ways

Augustus De Morgan on the history of mathematics (1865)

It is astonishing how strangely mathematicians talk of the Mathematics, because they do not know the history of their subject. By asserting what they conceive to be facts they distort its history . . . There is in the idea of every one some particular sequence of propositions, which he has in his own mind, and he imagines that that sequence exists in history . . .

Usually not the case!

Warning!

All this being said ...

Within the constraints imposed by this course (not least the need to fit it into 16 lectures), it will be all too easy to slip into a linear narrative of significant results

But this is not a disaster, provided we remain aware that it is happening. The lectures will provide much of the who, what, where, when; the classes will be where we can discuss the broader implications and historical questions. What can we learn from these mini historical episodes?

How do we organise the history of mathematics?

- periods (ancient, mediaeval, *n*th century, ...)
- places/cultures (Greece, Islam, Britain, ...)
- people (Archimedes, Newton, Euler, Hilbert, Noether, ...)
- topics (geometry, algebra, topology, probability, ...)
- sources (manuscripts, letters, books, journals, websites, ...)
- institutions (Royal Society, universities, LMS, ...)
- conferences (international congresses, local seminars, ...)

An outline of the course

Week 1: Introductory material: mathematics up to 1600

Week 2: Analytic geometry and the origins of calculus

Week 3: Newton's *Principia*; the further development of calculus

Week 4: Infinite series; the beginnings of rigour

Week 5: Algebra: from classical to modern

Week 6: Rigour in real analysis

Week 7: Complex analysis; linear algebra

Week 8: Geometry; number theory; historiography

Organisation by period

This course deals with (largely European) mathematics during the period 1600–1900

At different points of the course, we will consider particular places, people, and topics.

But if we were to divide up the course by century, we might see the following:

Organisation by period: 17th century

Topics:

- new notation
- analytic (co-ordinate) geometry
- calculus
- infinite series
- mathematics applied to the physical world

People: Descartes, Fermat, Wallis, Newton, Leibniz, Huygens, l'Hôpital, ...

Organisation by period: 18th century

Topics:

- many applications of (and some problems with) calculus
- applications (and problems) of infinite series
- developments in algebra and number theory
- mathematics applied to the physical world

People: Bernoullis, Euler, d'Alembert, de Moivre, Laplace, Lagrange, ...

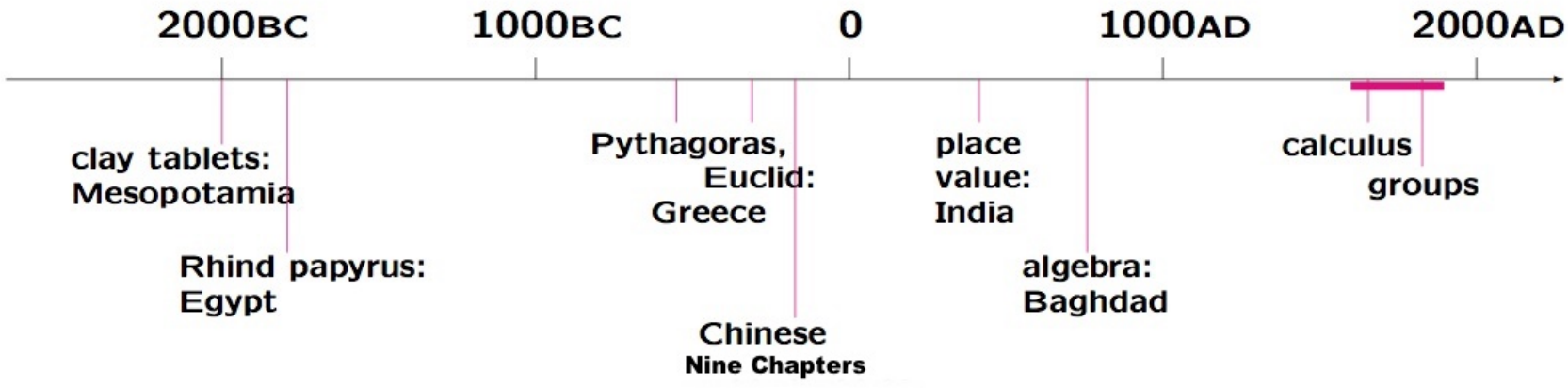
Organisation by period: 19th century

Topics:

- from calculus to analysis
- development of complex analysis
- rise of abstract algebra
- beginnings of linear algebra
- non-Euclidean geometry
- beginnings of axiomatisation

People: Gauss, Fourier, Bolzano, Cauchy, Abel, Galois, Dirichlet, Cayley, Dedekind, Cantor, ...

A timeline


2000BC
1000BC
0
1000AD
2000AD
clay tablets: Mesopotamia
Rhind papyrus: Egypt
Pythagoras, Euclid: Greece
Chinese Nine Chapters
place value: India
algebra: Baghdad
calculus
groups

But what about non-Western mathematics?

Much of the world's present-day underlying mathematical culture — the way of doing it (e.g., notation, structure of arguments), of publishing it, and so on — is of European origin, and it is worth bearing in mind why.

The export of European mathematics to the rest of the world has been going on for centuries — for example, at the end of the 16th century, Jesuit missionaries began to introduce European mathematics into China, where it soon supplanted local mathematical traditions.

In the period 1600–1900, most of the parts of the world that had a culture of mathematics that went beyond arithmetic were doing mathematics in a European style — a legacy of imperialism and colonialism.

As we go through the course, please keep in mind that the way in which we do mathematics now is neither the only way to do it nor necessarily the 'best' way, and that what is 'best' might differ depending on the context.

Wasan

An example of a mathematical tradition that was significantly from the European one was wasan 和算 (Japanese mathematics) which flourished during the Edo Period (1603–1867)

See Tsukane OGAWA, 'A review of the history of Japanese mathematics', *Revue d'histoire des mathématiques* **7** (2001) 137–155

and also Japanese Mathematics in the Edo Period

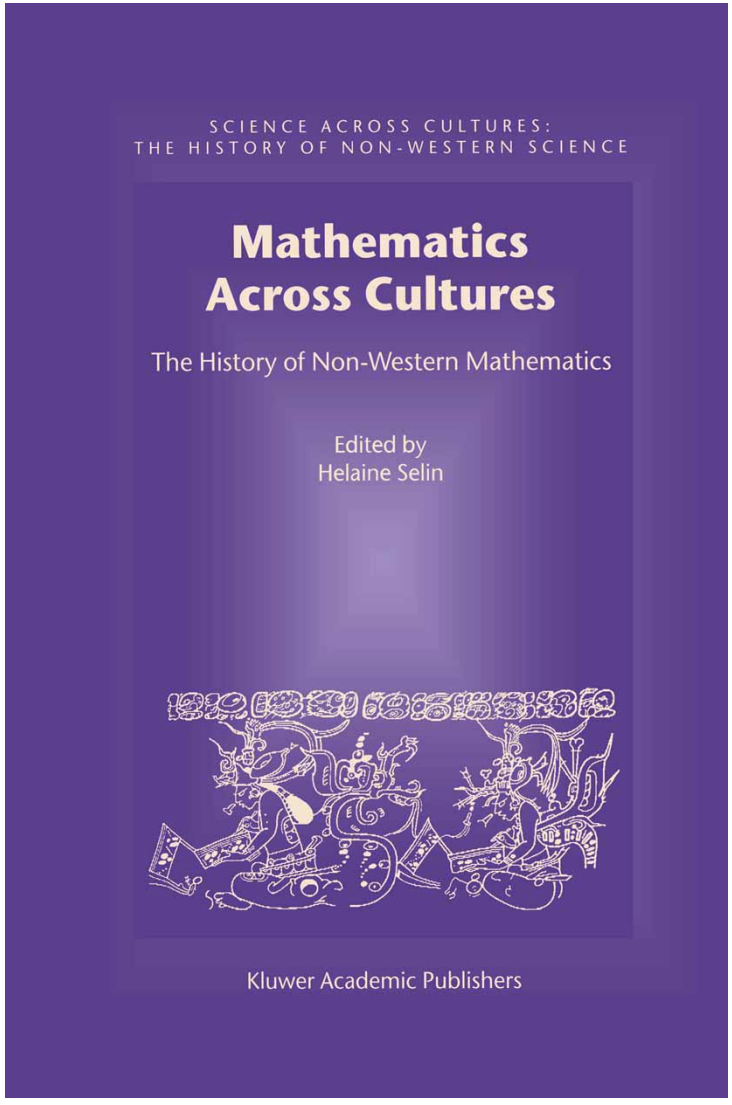
Other non-European mathematics

In this course, we will see other instances of non-European mathematics that are important for our story; for example:

- solution of systems of linear equations in the *Jiǔzhāng Suànshù* (2nd century BC, China) [week 7]
- solution of polynomial equations by al-Khwārizmī (9th century Baghdad) [week 5]
- study of infinite series by the 'Kerala School' (14th–16th-century India) [week 4]

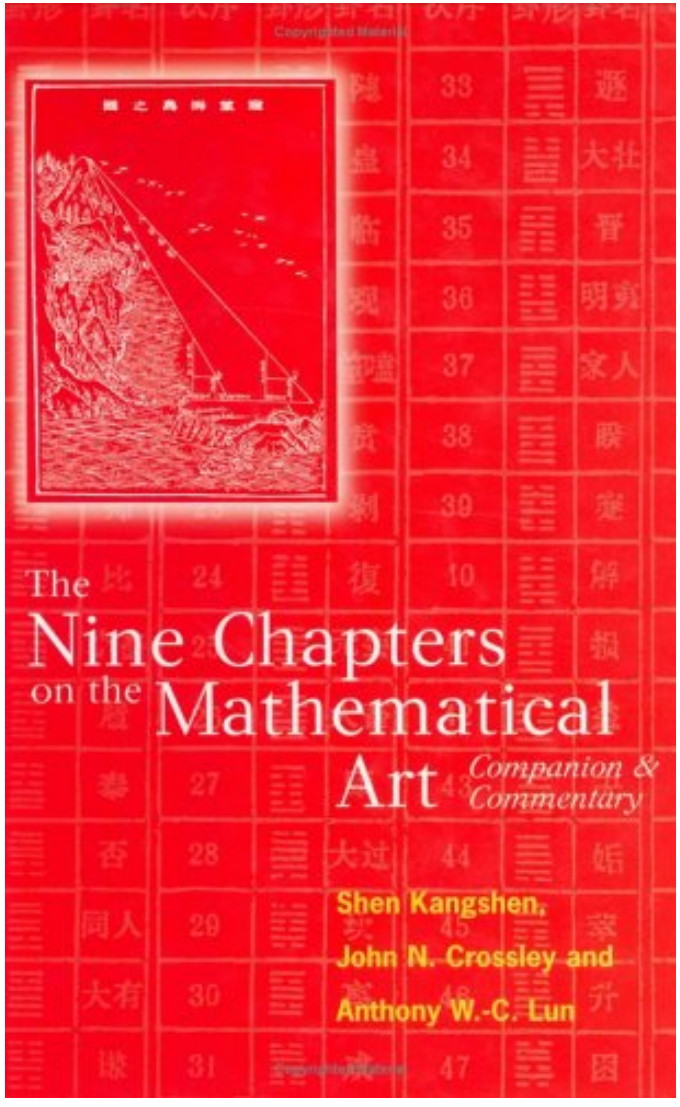
But we will not be systematic in our treatment of non-European mathematics

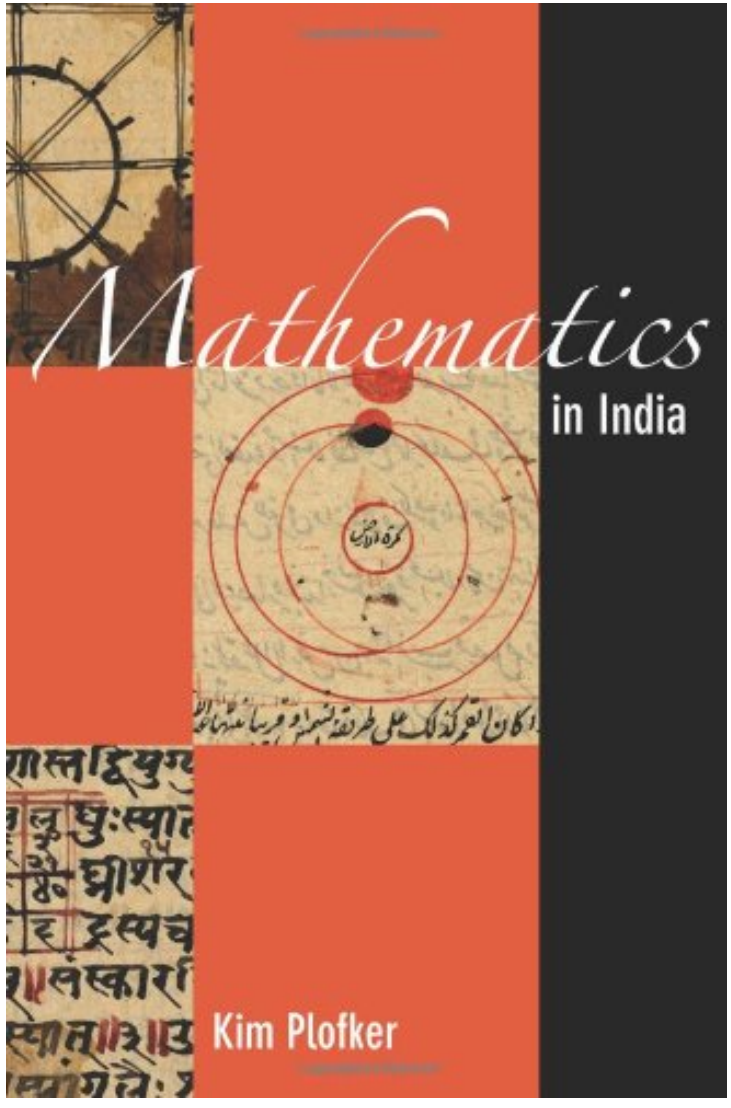
On non-European mathematics



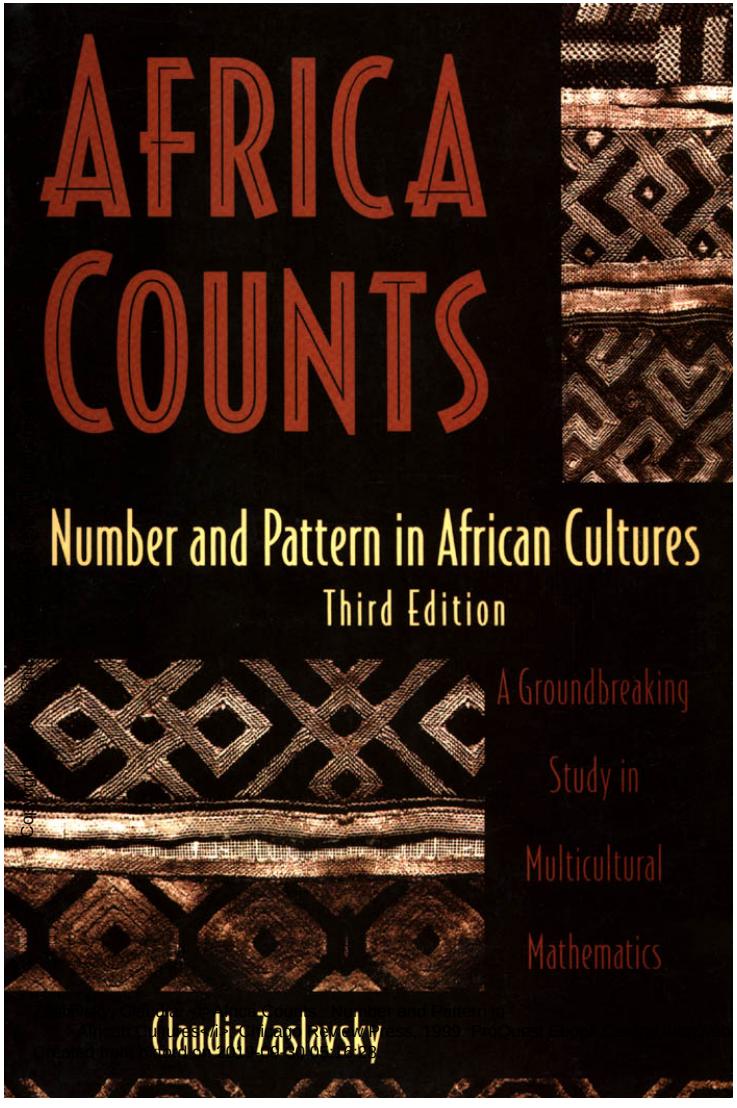
Helaine Selin (ed.), *Mathematics across Cultures: The history of non-Western mathematics*, Kluwer Academic Publishers, 2000

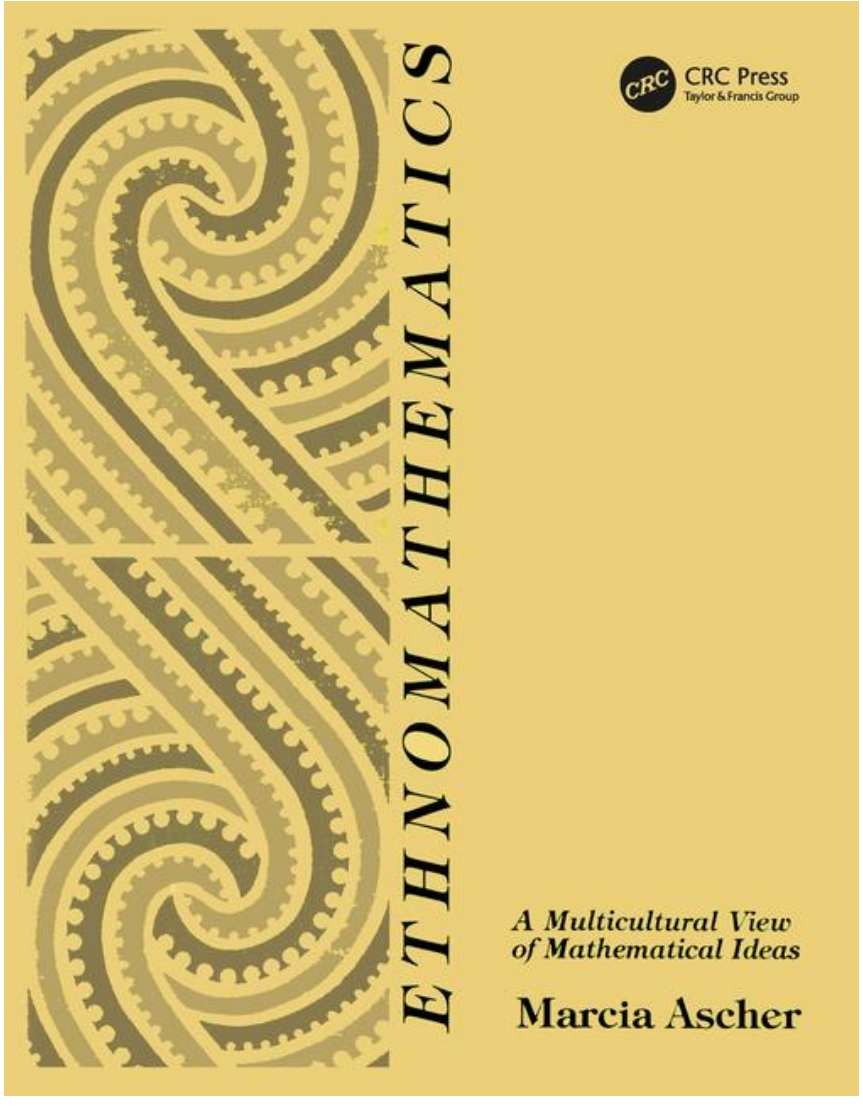
On non-European mathematics





On non-European mathematics





Administrative matters, MT

- Lectures: Tuesdays, 11:00–13:00, with a break in the middle, from week 2 onwards, lectures will be in L1
  (lecture slides to be posted online week by week)

- Classes: to be held on Wednesdays in weeks 3, 5, 6, 7, 13:30–15:00 and 15:00–16:30, in C4

- Work: 1,000-word essay and preparation of a discussion topic for each class — essays to be submitted via the course webpage by 9am on the Mondays of weeks 3, 5, 6, 7

- Sheets: details of weekly reading, and of essay and discussion topics, can be found on course webpage

- Assessment: written paper in TT 2023

Administrative matters, HT

Topic: Isaac Newton's *Arithmetica universalis*

- A reading course with 'seminars' in weeks 1–8
- In-depth study of original works by Newton, Maclaurin, Euler, and others
- Two or three essays of 2,000 words each, and preparation of a discussion topic for each class
- Assessment: 3,000-word essay, topic revealed in week 7, to be submitted by 12 noon on Monday, week 10

Advice on taking notes and writing essays

- See the notes on the course webpage
- Pay particular attention to the sections on
  - citing sources,
  - bibliographies, and
  - plagiarism
- The importance of clear and accurate citations will be stressed throughout the course — these serve the same purpose as proofs in mathematical arguments

Taking notes

From reading:

- *background* reading is for information, not examination: it is important, but don't spend too long on it
- read the material (at least) twice
  - on the first reading, try to get a general feel for the material, its meaning and significance
  - on the second, take notes (see the online guidance)

From lectures:

- Don't try to take down every detail
- Instead, read ahead, listen, think, ask questions

The lectures and the reading will cover some of the same ground but are designed to be complementary

Recommended reading: the main texts

Jacqueline Stedall, *Mathematics emerging: a sourcebook 1540–1900*, Oxford University Press, 2008 [available via SOLO]

and

Victor Katz, *A history of mathematics: an introduction*, 3rd edition, Addison-Wesley, 2009

(College libraries may have earlier editions of the latter, but these do not differ significantly in content as far as this course is concerned.)

Recommended reading: other useful books

Jacqueline Stedall, *The history of mathematics: a very short introduction*, Oxford University Press, 2012 [available via SOLO]

Benjamin Wardhaugh, *How to read historical mathematics*, Princeton University Press, 2010 [available via SOLO]

John Fauvel and Jeremy Gray, *The history of mathematics: a reader*, Macmillan/Open University, 1987

June Barrow-Green, Jeremy Gray and Robin Wilson, *The history of mathematics: a source-based approach*, vol. 1, MAA Press, 2019; vol. 2, 2022

Further books (usually on specific topics) will be cited throughout the course

Recommended reading: other useful resources

Some biographical resources:

(Complete) Dictionary of Scientific Biography (*DSB*): available via SOLO

Oxford Dictionary of National Biography (*ODNB*): available via SOLO

There are many other general histories of mathematics available — you are encouraged to read widely, but please read critically

BO1 History of Mathematics
Lecture II
Dissemination and development
(600 BC – AD 1600)

MT 2022 Week 1

Summary

- Influence of the ancient world
- The European Renaissance (15th and 16th centuries)
- Rediscovery and transmission of ancient texts
- The 16th century
- A case study: Napier's invention of logarithms 1614

Ancient influences on early modern European mathematics

(Early modern = roughly 1400–1800)

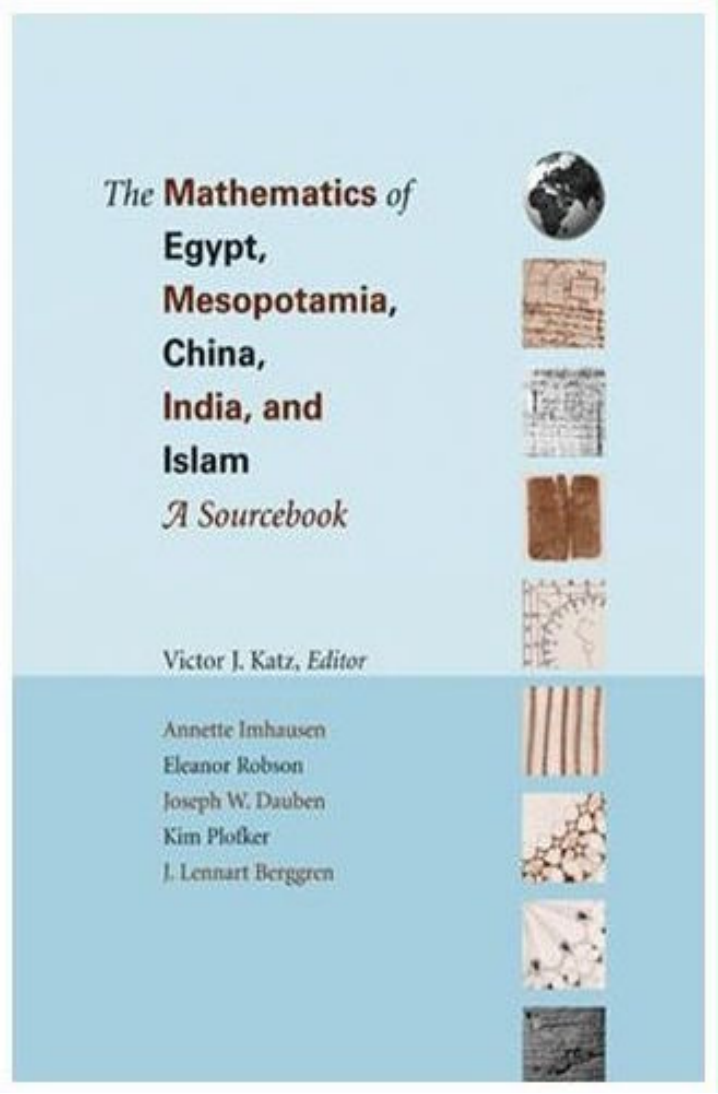
The prior mathematical accomplishments of

- China
- India
- Mesopotamia
- Egypt
- and most other places

were largely unknown in Europe until the 19th century

The single greatest influence on early modern European mathematics was the mathematics of ancient Greece, with that of mediaeval Islam (a little later) coming a close second

Ancient origins of mathematics



On the ancient origins of mathematics, see:

Victor J. Katz (ed.), *The mathematics of Egypt, Mesopotamia, China, India, and Islam: a sourcebook*, Princeton University Press, 2007

Ancient Greek mathematics

Earliest origins of Greek mathematics in 6th century BC

But what do we mean by 'Greek'?

500 BC – 300 BC	Collection of city-states in Greece
300 BC – AD 500	Greek-speaking peoples around the Mediterranean, especially in Alexandria

Some major figures of 'Greek' mathematics

Pythagoras	Samos (Greece)?	c. 600 BC
Euclid	Alexandria (Egypt)?	c. 300 (or 250?) BC
Archimedes	Syracuse (Sicily)	c. 250 BC
Apollonius	Perga (Turkey)	c. 180 BC
Diophantus	Alexandria (Egypt)	c. AD 200

Euclid's *Elements*

The 'elements of geometry' in 13 books, compiled around 300 (250?) BC from existing geometrical knowledge

Books I–VI	plane geometry	points, lines, angles, circles, ...
Books VII–X	properties of numbers	odd, even, square, triangular, prime, perfect, ...
Books XI–XIII	solid geometry	cube, tetrahedron, icosahedron, ...

David Joyce's Java version of Euclid's *Elements*

Oliver Byrne's colour version of the first six books

Euclid's *Elements*, book I

23 definitions: point, line, surface, angle, circle, ...

5 postulates: what one can do

e.g. a straight line may be drawn between any two points;
a circle may be drawn with given centre and radius

5 'common notions': how one may reason

e.g. if equals are added to equals, then the wholes are equal

48 propositions: each built only on what has gone before

The influence of Euclid's *Elements*

HUGE influence on Western mathematics:

- hundreds of editions and translations from renaissance onwards
- basis of mathematics teaching in schools until c. 1960
- style: definitions—axioms—theorems—proofs
- status of 'Parallel Postulate' led to much investigation and, ultimately, non-Euclidean geometries
- problems of 'ruler and compass' construction inspired much investigation and many new discoveries
- wider cultural importance: http://readingeuclid.org/

Other Greek authors

Archimedes	d. 212 BC:	method of exhaustion and much else
Apollonius	c. 180 BC:	conic sections
Diophantus	c. AD 250:	*Arithmetica* in 13 books (number problems)

Also had HUGE influence on Western mathematics

Remnants of the collapse of the ancient world

in Greek:	manuscripts preserved at Constantinople and in libraries or collections around the Mediterranean
in Latin:	writings by Boethius (c. 480–524) on philosophy, arithmetic, geometry, music

The spread of Islam and Islamic learning

632–732:	Islam spreads throughout Middle East, north Africa, and into Spain and Portugal
c. 820:	*Bayt al-Ḥikma*, the House of Wisdom, founded in Baghdad under Caliph al-Ma'mūn; it became a centre for translation into Arabic from Greek, Persian, Sanskrit
c. 825:	al-Khwārizmī active in Baghdad
9th century:	texts on arithmetic, algebra, astronomy reach Spain
12th century:	translations from Arabic to Latin

The mid-Renaissance (15th and 16th centuries)

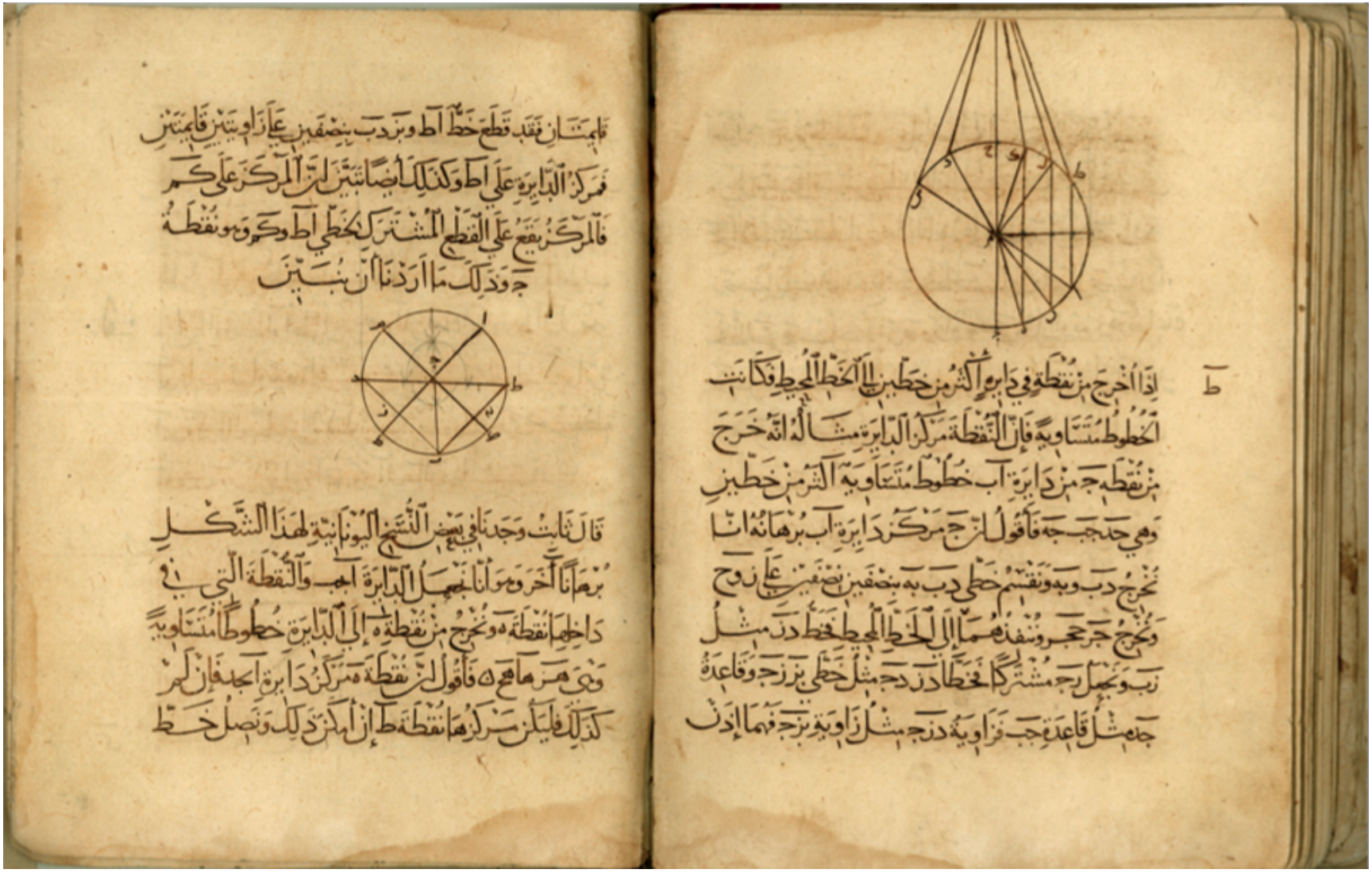
Classical mathematical texts more widely available due to:

- rediscovery of manuscripts
- revival of knowledge of Greek
- transmission of otherwise lost texts via Arabic versions
- (Western) invention of printing (Gutenberg, c. 1436)

Euclid's *Elements*: transmission history

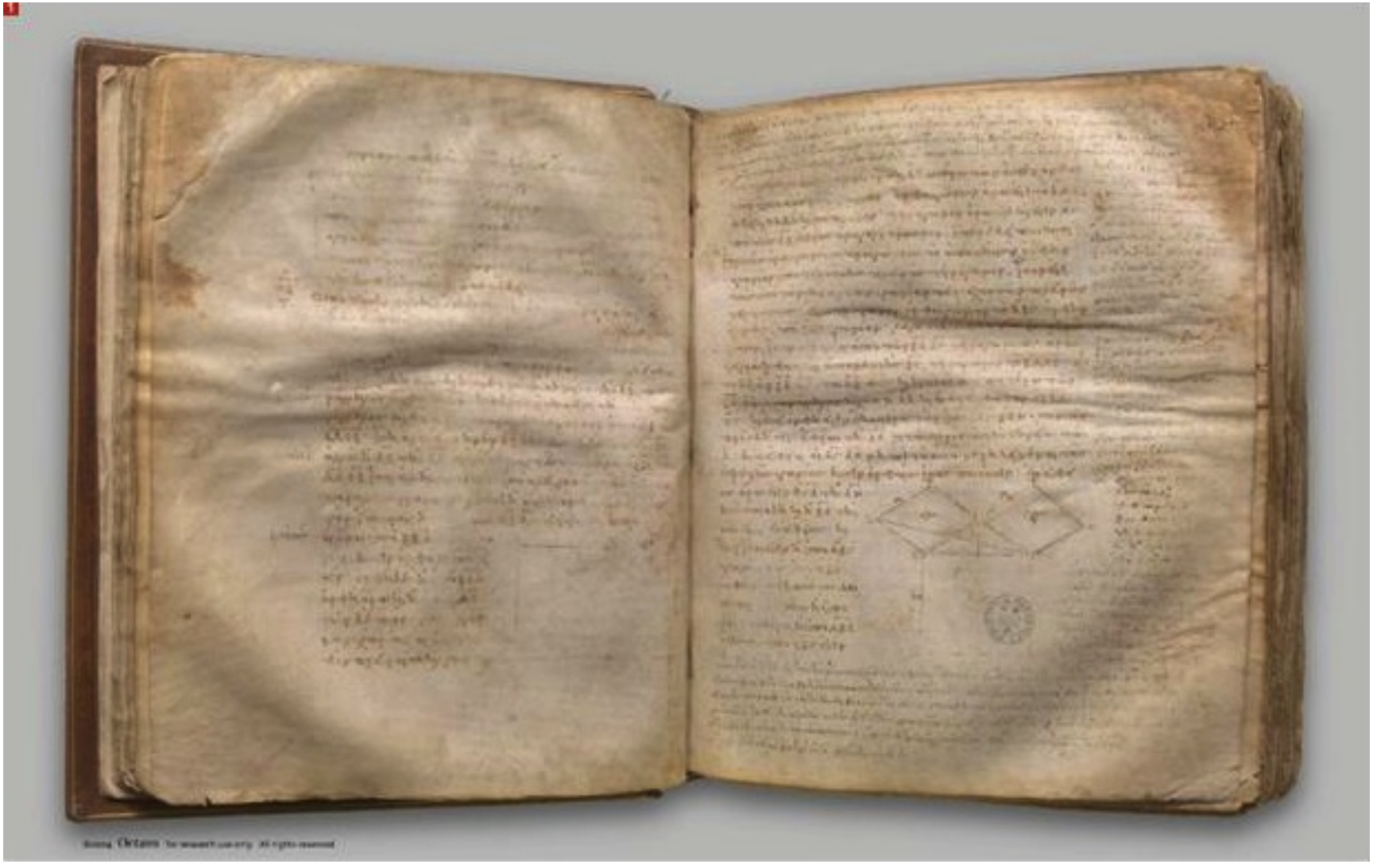
- commentaries written by Pappus (c. AD 320), Theon (c. AD 380), Proclus (c. AD 450)
- a few propositions in Boethius (c. AD 500)
- copies in Greek (earliest from Constantinople, AD 888)
- many translations or commentaries in Arabic (AD 750–1250)
- mediaeval translations from Arabic to Latin: Adelard of Bath (1130), Robert of Chester (1145), Gerard of Cremona (mid-12th century)
- printed editions in Latin or Greek from 1482 onwards

Euclid in Arabic



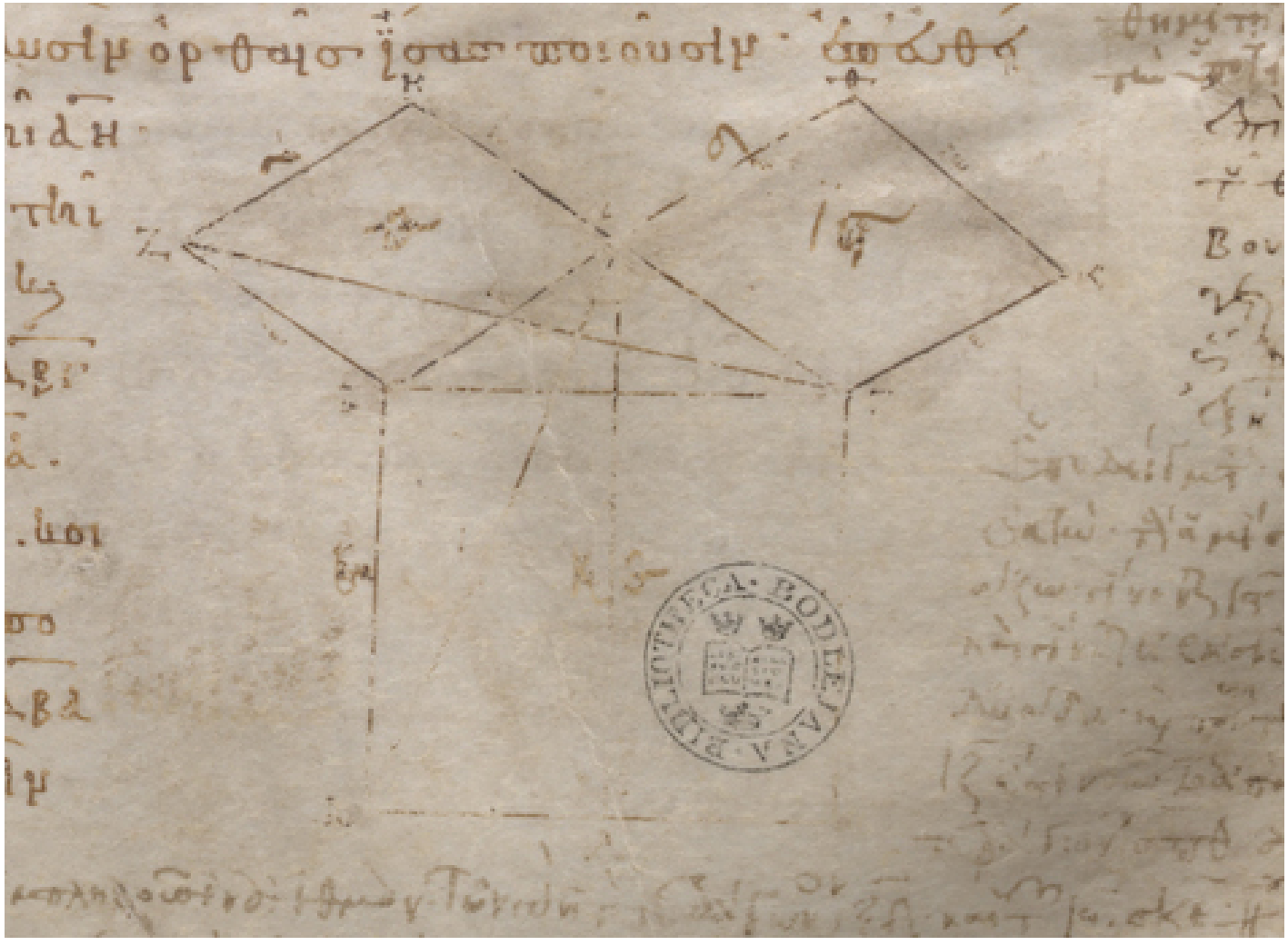
Translated from the Greek by Ishaq ibn Hunayn, AD 1466

Euclid I.47 from Bodleian ms. dated 888



Whole manuscript is digitised:
http://www.claymath.org/library/historical/euclid/

Euclid I.47 from Bodleian ms. dated 888



http://www.claymath.org/library/historical/euclid/files/elem.1.47.html

Treatises by Archimedes: transmission history

- quoted or explained by Pappus (c. 320 AD), Theon (c. 380 AD), Eutocius (c. 520 AD)
- 6th-century Byzantine 'collected works' (Isidore of Miletus)
- several translations of individual treatises into Arabic
- translations from Arabic into Latin
- a new find in the twentieth century: www.archimedespalimpsest.org/

Apollonius' *Conics* (c. 180 BC): transmission history

- Books I–IV survived in Greek
- Books V–VII survived only in Arabic
- Book VIII is lost, known only from commentaries
- early (Latin) printed edition, 1566

(See: *Mathematics emerging*, §1.2.4.)

16th century change

New forces at work in the 16th century:

- global exploration
- growth of international commerce
- new technology (in printing, shipping, military engineering, instrumentation, etc.)

A case study of a text from 1614

Napier's invention of logarithms:

- what did 17th-century mathematics look like?
- how can we begin to read historical texts?

Napier's definition of a logarithm (of a sine)

The **Logarithme** *therefore of any sine is a number very neerely expressing the line, which increased equally in the meane time, whiles the line of the whole sine decreased proportionally into that sine, both motions being equal-timed, and the beginning equally swift.*

?

Context, content, significance

Context:	who? when? where? why?
Content:	what is it about? how is it written?
Significance:	why did/does it matter?
Historical Significance:	what new insight does this text offer us?

Context — who?

John Napier (1550–1617), Merchiston, Scotland

Scottish landowner with interests in:

- mining
- calculating aids
- astrology/astronomy
- The Book of Revelation

See *Oxford Dictionary of National Biography*:
http://www.oxforddnb.com/view/article/19758

Context — why?

From Napier's preface to the English translation of 1616:

> *Seeing there is nothing (right well-beloved Students of the Mathematics) that is so troublesome to mathematical practice, nor that doth more molest and hinder calculators, than the multiplications, divisions, square and cubical extractions of great numbers, which besides the tedious expense of time are for the most part subject to many slippery errors, I began therefore to consider in my mind by what certain and ready art I might remove those hindrances.*

Context — why?

Inspired by the 16th-century technique of prosthaphaeresis:

the use of trigonometric identities such as

$$\cos x \cos y = \frac{1}{2}\left[\cos(x+y) + \cos(x-y)\right]$$

$$\sin x \sin y = \frac{1}{2}\left[\cos(x-y) - \cos(x+y)\right]$$

to convert multiplication into addition.

Context — in what form, and in which language?

Original Latin text of 1614:

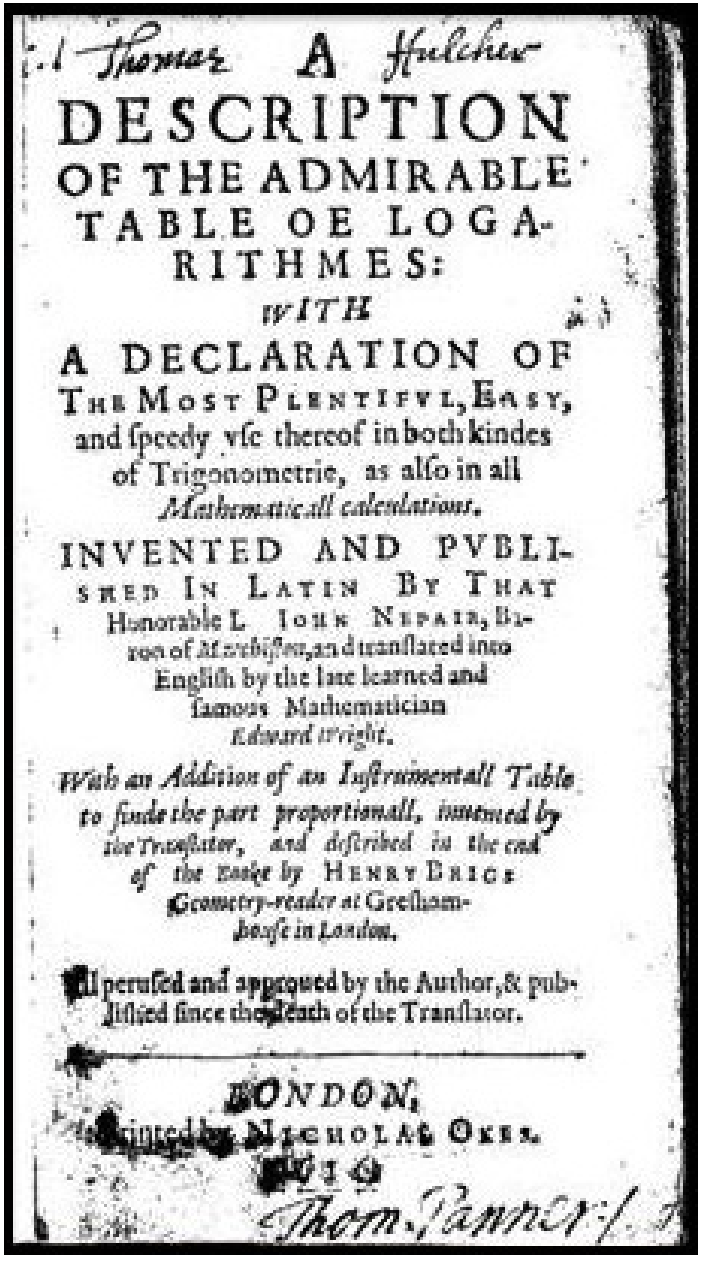
Mirifici logarithmorum canonis descriptio

translated into English by Edward Wright in 1616 as

A description of the admirable table of logarithms

Scanned text available via SOLO

Napier's 1616 title-page decoded



A DESCRIPTION
OF THE ADMIRABLE
TABLE OE LOGA-
RITHMES:
WITH
A DECLARATION OF
THE MOST PLENTIFVL, EASY,
and speedy vse thereof in both kindes
of Trigonometrie, as also in all
Mathematicall calculations.
INVENTED AND PVBLI-
SHED IN LATIN BY THAT
Honorable L IOHN NEPAIR, Ba-
ron of Merchiston, and translated into
English by the late learned and
famous Mathematician
Edward Wright.
With an Addition of an Instrumentall Table
to finde the part proportionall, invented by
the Translator, and described in the end
of the Booke by HENRY BRIGS
Geometry-reader at Gresham-
house in London.
All perused and approued by the Author, & pub-
lished since the death of the Translator.
LONDON,
Printed by NICHOLAS OKES.
1616
Thom. Panner

Inventor:
- John Napier (1550–1617)

Translator:
- Edward Wright (?1558–1615) (interests: navigation, charts and tables)

Additional material:
- Henry Briggs (1561–1630) Gresham Professor of Geometry, later Savilian Professor of Geometry at Oxford (interests: navigation)

Printer:
- Nicholas Okes

Readers:
- Thomas Hulcher,
- Thomas Panner

Napier's logarithms: content

Recall:

> *The* **Logarithme** *therefore of any sine is a number very neerely expressing the line, which increased equally in the meane time, whiles the line of the whole sine decreased proportionally into that sine, both motions being equal-timed, and the beginning equally swift.*

Napier's logarithms

peare by the 19 *Prop.* 5. and 11. *Prop.* 7, *Euclid.*

3 *Def.* *Surd quantities, or vnexplicable by number, are said to be defined, or expressed by numbers very neere, when they are defined or expressed by great numbers which differ not so much as one vnite from the true value of the Surd quantities.*

As for example. Let the semidiameter, or whole sine be the rational number 10000000 the sine of 45 degrees shall be the square root of 50,000,000,000,000, which is surd, or irrationall and inexplicable by any number, & is included between the limits of 7071067 the lesse, and 7071068 the greater: therfore, it differeth not an vnite from either of these. Therefore that surd sine of 45 degrees, is said to be defined and expressed very neere, when it is expressed by the whole numbers, 7071067, or 7071068, not regarding the fractions. For in great numbers there ariseth no sensible error, by neglecting the fragments, or parts of an vnite.

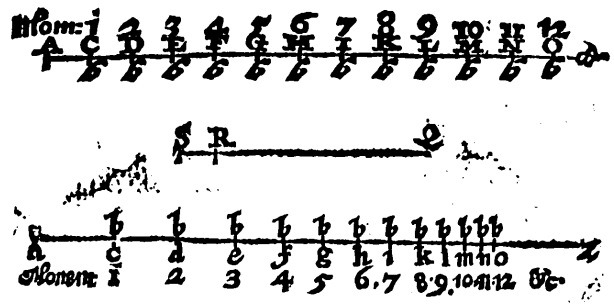
4 *Def.* *Equall-timed motions are those which are made together, and in the same time.*

As in the figures following, admit that B be moued from A to C, in the same time, wherin *b* is moued from *a* to *c* the right lines A C & *a c*, shall be sayd to be described with an equall-timed motion.

5 *Def.* *Seeing that there may bee a slower and a swifter motion giuen then any motion, it shall necessarily follow, that there may be a motion giuen of equall swiftnesse to any motion (which wee define to be neither swifter nor slower.)*

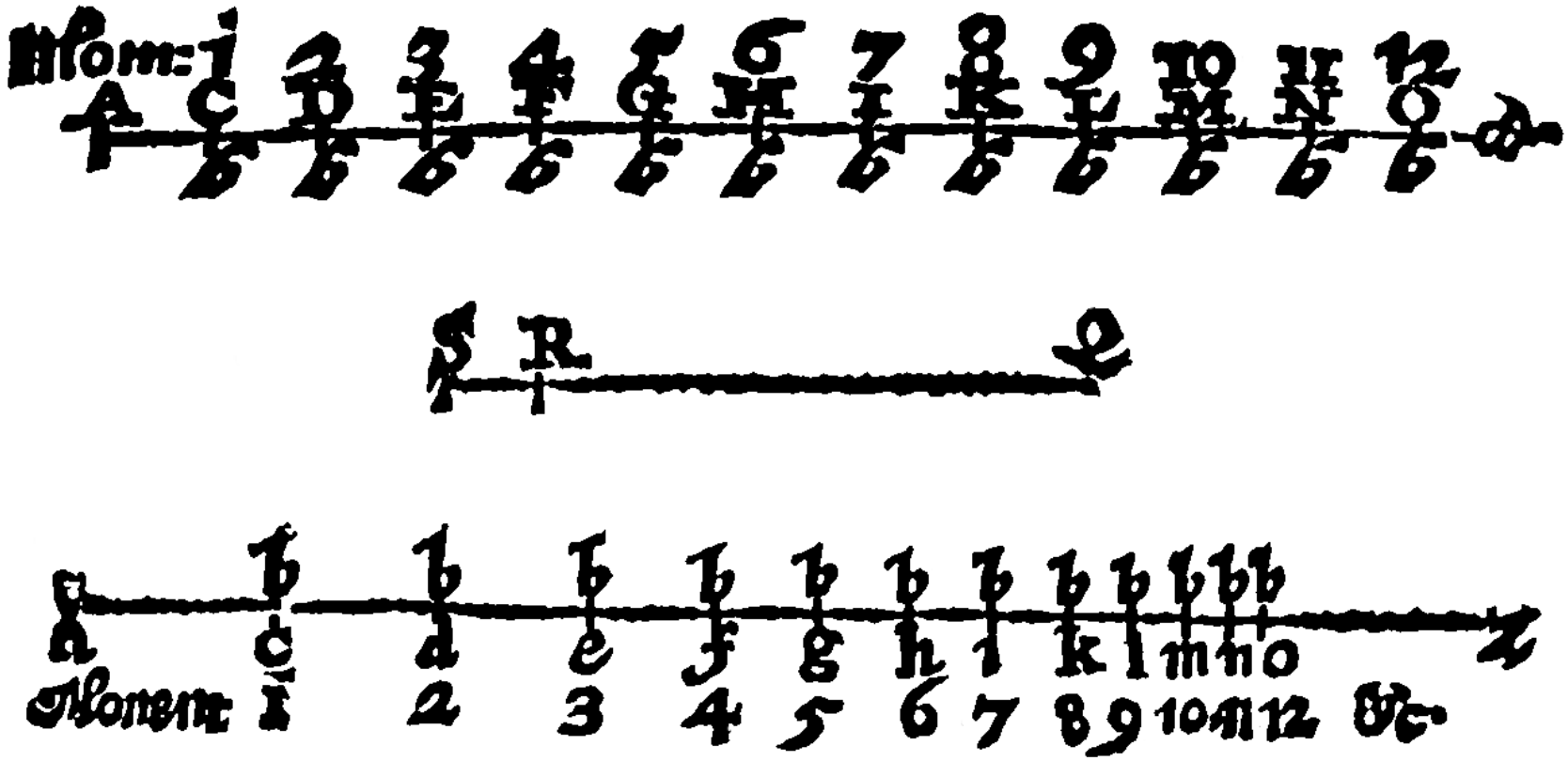
6 *Def.* The Logarithme *therfore of any sine is a number very neerely expressing the line, which increased*

sed equally in the meane time, whiles the line of the whole sine decreased proportionally into that sine, both motions being equal-timed, and the beginning equally swift.

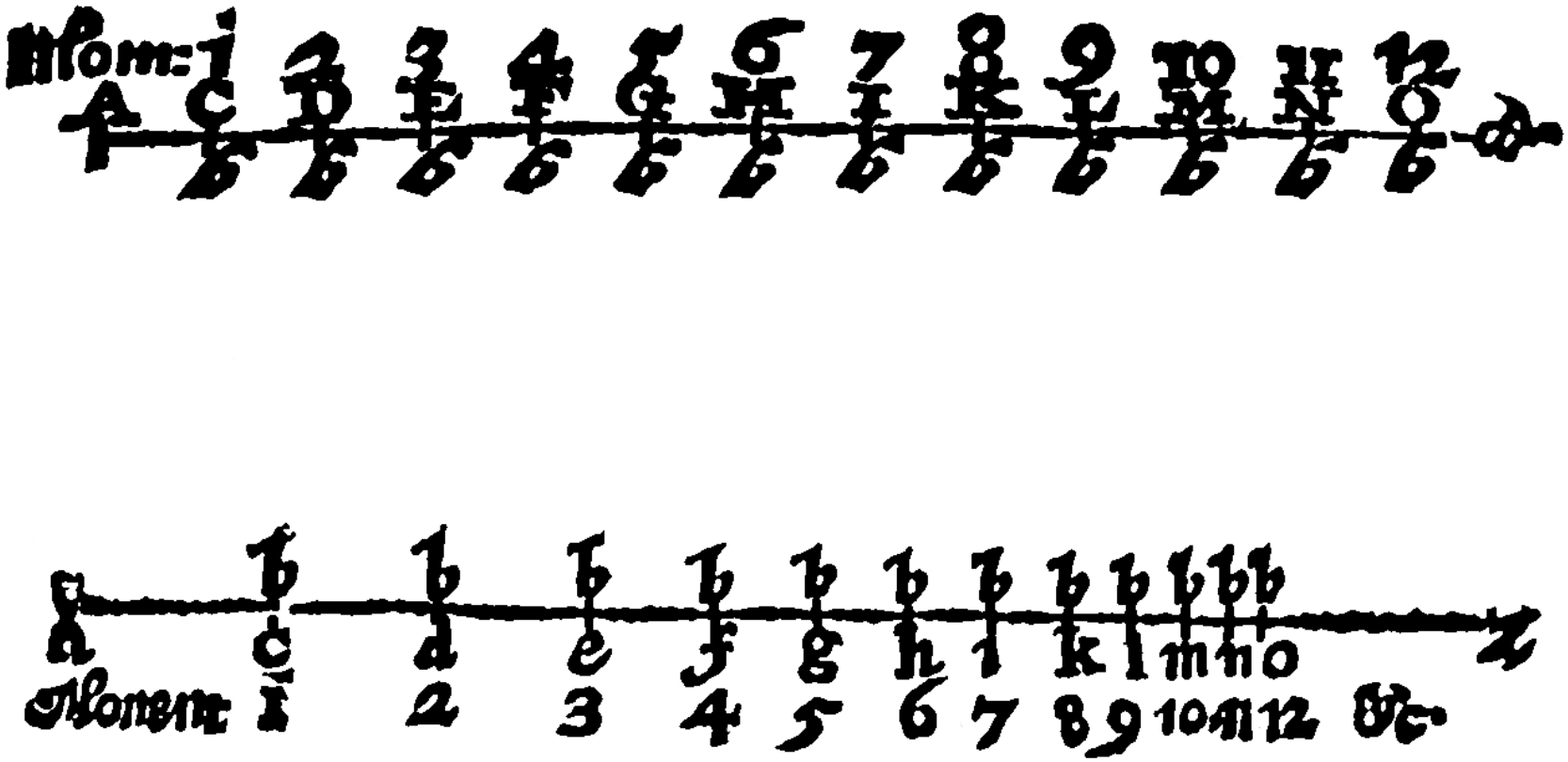


As for example. Let the 2 figures going afore bee here repeated, and let B bee moued alwayes, and euery where with equall, or the same swiftnesse wherewith *b* beganne to bee moued in the beginning, when it was in *a*. Then in the first moment let B proceed from A to C, and in the same time let *b* moue proportionally from *a* to *c*, the number defining or expressing A C shal be the *Logarithme* of the line, or sine *c* Z. Then in the second moment let B bee moued forward from C to D. And in the same moment or time let *b* be moued proportionally from *c* to *d*, the number defininig A D, shall bee the *Logarithme* of the sine *d* Z. So in the third moment let B go forward equally from D to E, and in the same moment let *b* be moued forward proportionally from *d* to *e*, the number expressing A E the *Logarithme* of the sine *e* Z. Also in the fourth moment, let B proceed

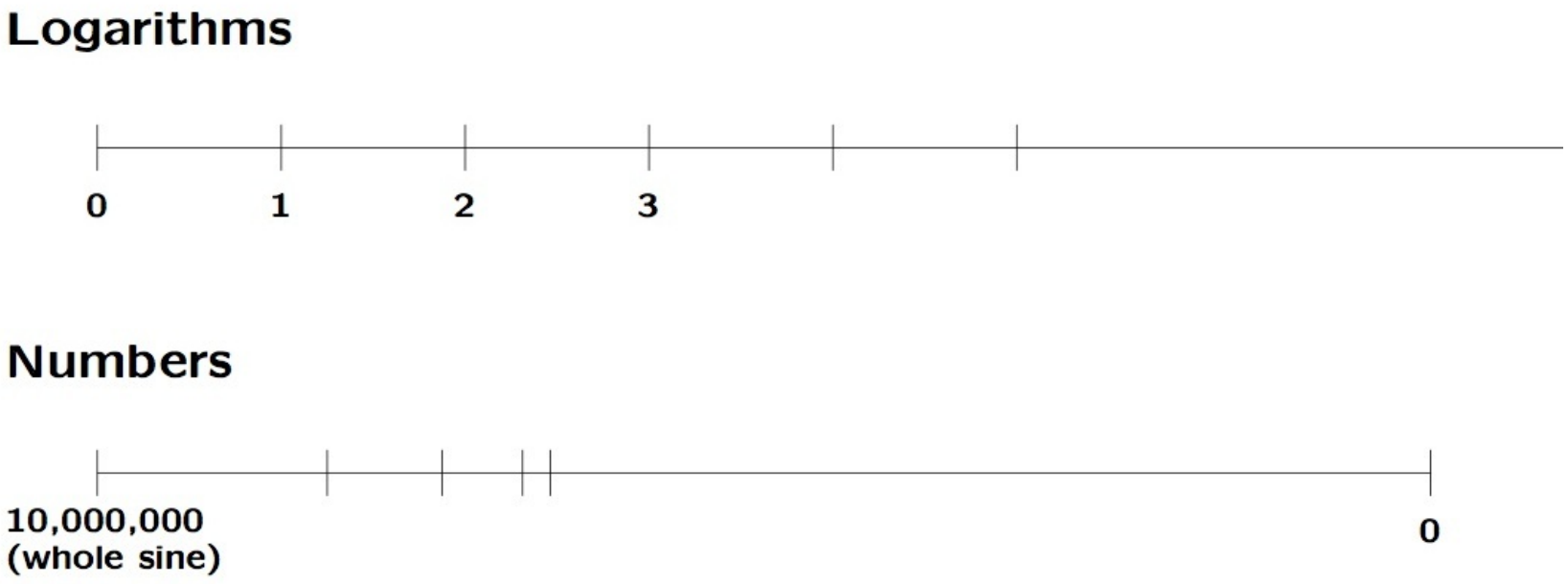
Napier's logarithms



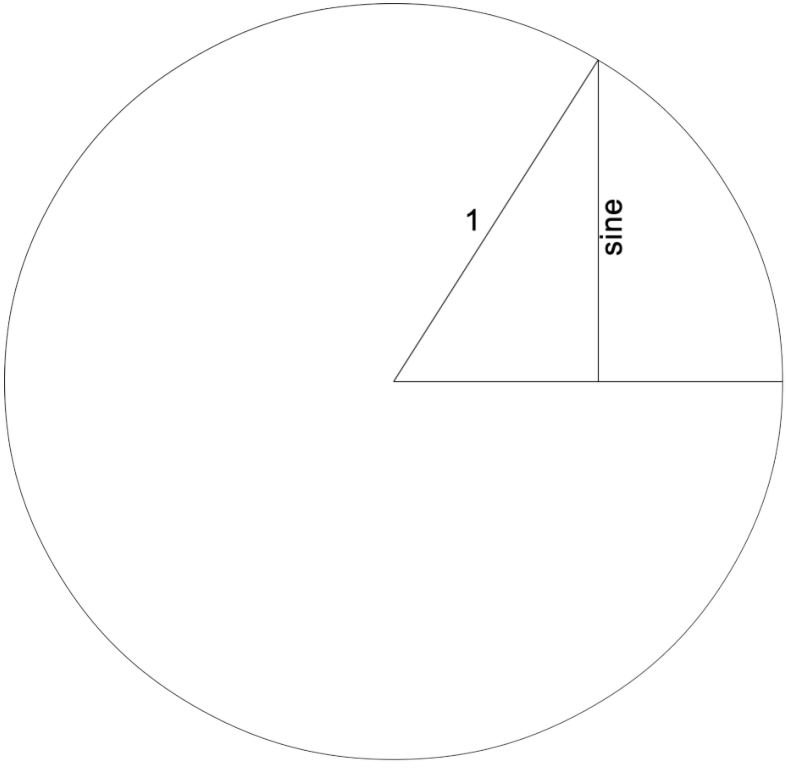
Napier's logarithms



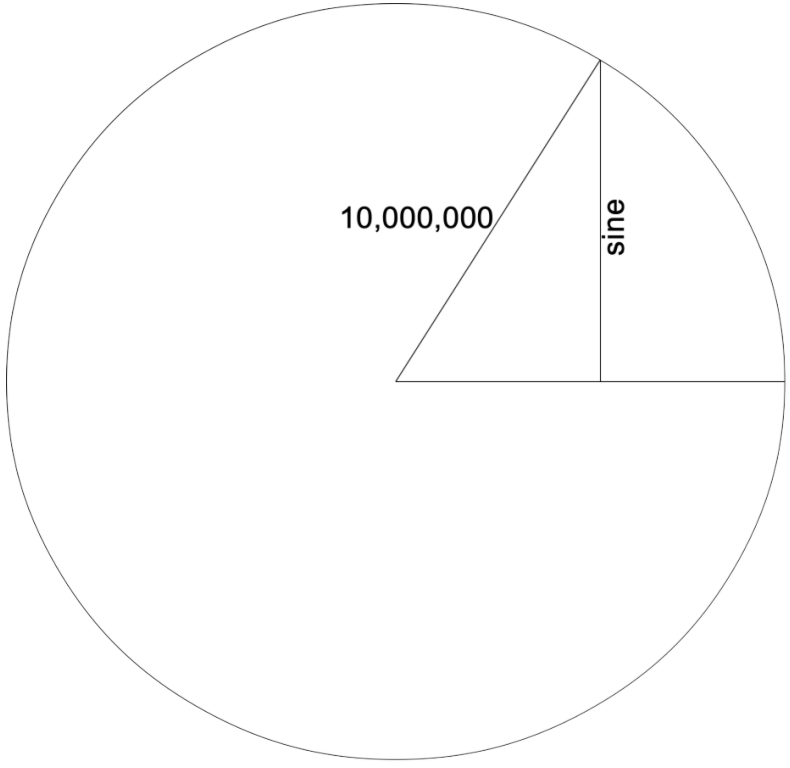
Napier's logarithms



Napier's logarithms

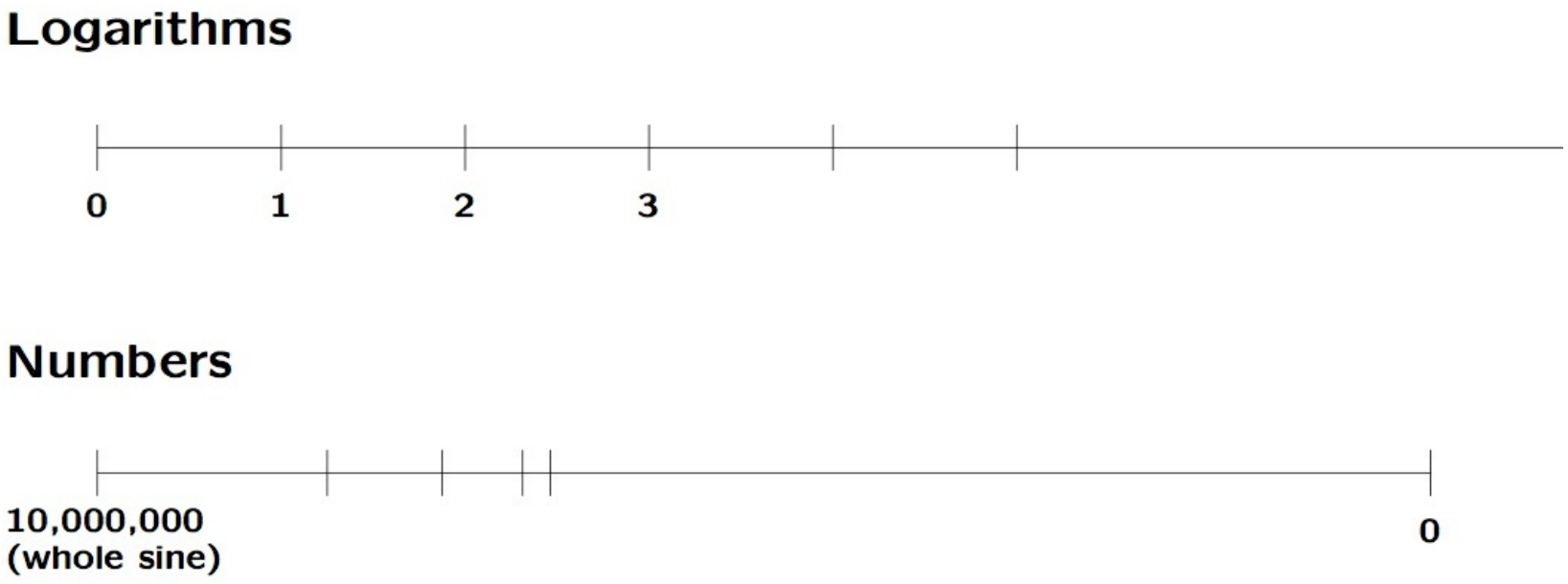


Sine of angle at centre varies between 0 and ± 1 as the labelled radius sweeps around the circle



Sine of angle at centre varies between 0 and $\pm 10,000,000$ as the labelled radius sweeps around the circle

Napier's logarithms



Napier's logarithms (1614)

In modern terms (i.e., not Napier's):

$$\text{if } y = 10^7\left(1 - 10^{-7}\right)^x, \text{ then } \operatorname{Nap\,log}\, y = x$$

$\operatorname{Nap\,log}\, 10^7 = 0$, $\operatorname{Nap\,log}\, 0$ is infinite, $\operatorname{Nap\,log}\, 1 = 161,180,956$

$$\operatorname{Nap\,log}\left(\frac{p \times q}{10^7}\right) = \operatorname{Nap\,log}\, p + \operatorname{Nap\,log}\, q$$

$$\operatorname{Nap\,log}\,(p \times q) = \operatorname{Nap\,log}\, p + \operatorname{Nap\,log}\, q - \operatorname{Nap\,log}\, 1$$

Note that $\operatorname{Nap\,log}\, x = 10^7 \ln\left(\frac{10^7}{x}\right)$

No notion of base, although Nap log 'nearly' has base $\frac{1}{e}$ — see:
Robin Wilson, *Euler's Pioneering Equation*, OUP, 2019, p. 101

Modifications by Napier and Briggs (1617)

Definition revised to remove the need to subtract $\mathrm{Nap}\log 1$

'Briggsian' logarithms have base 10 and $\mathrm{Log}\,1 = 0$, so that

$$\mathrm{Log}\,(p \times q) = \mathrm{Log}\,p + \mathrm{Log}\,q$$

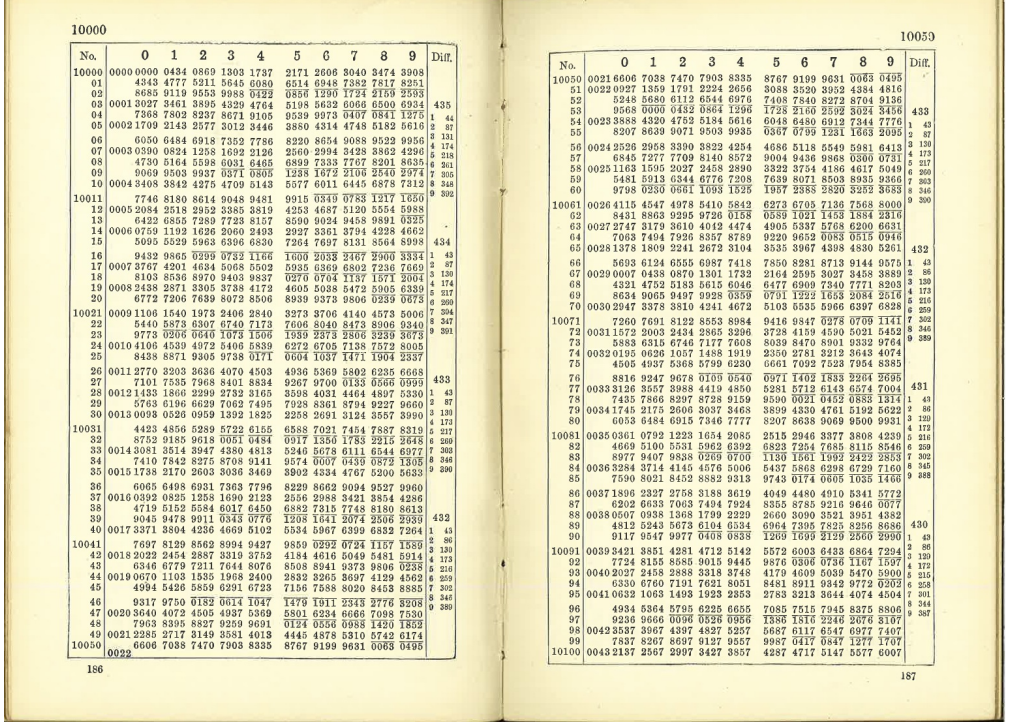
Briggs produced *Logarithmorum chilias prima* (*The first thousand logarithms*) in 1617, followed by his *Arithmetica logarithmica* in 1624, which contained logarithms of 1 to 20,000 and 90,000 to 100,000, all to 14 decimal places (calculated by hand); the gap in the table was filled by Adriaan Vlacq in 1628

Napier's logarithms

One last time:

> *The* **Logarithme** *therefore of any sine is a number very neerely expressing the line, which increased equally in the meane time, whiles the line of the whole sine decreased proportionally into that sine, both motions being equal-timed, and the beginning equally swift.*

Significance



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02		8685	9119	9553	9988	0422	0856	1290	1724	2159	2593	
03	0001 3027	3461	3895	4329	4764	5198	5632	6066	6500	6934	435
04		7368	7802	8237	8671	9105	9539	9973	0407	0841	1275	1 44
05	0002 1709	2143	2577	3012	3446	3880	4314	4748	5182	5616	2 87
06		6050	6484	6918	7352	7786	8220	8654	9088	9522	9956	3 131
07	0003 0390	0824	1258	1692	2126	2560	2994	3428	3862	4296	4 174
08		4730	5164	5598	6031	6465	6899	7333	7767	8201	8635	5 218
09		9069	9503	9937	0371	0805	1238	1672	2106	2540	2974	6 261
10	0004 3408	3842	4275	4709	5143	5577	6011	6445	6878	7312	7 305
											8 348
10011		7746	8180	8614	9048	9481	9915	0349	0783	1217	1650	9 392
12	0005 2084	2518	2952	3385	3819	4253	4687	5120	5554	5988	
13		6422	6855	7289	7723	8157	8590	9024	9458	9891	0325	
14	0006 0759	1192	1626	2060	2493	2927	3361	3794	4228	4662	
15		5095	5529	5963	6396	6830	7264	7697	8131	8564	8998	434
16		9432	9865	0299	0732	1166	1600	2033	2467	2900	3334	1 43
17	0007 3767	4201	4634	5068	5502	5935	6369	6802	7236	7669	2 87
18		8103	8536	8970	9403	9837	0270	0704	1137	1571	2004	3 130
19	0008 2438	2871	3305	3738	4172	4605	5038	5472	5905	6339	4 174
20		6772	7206	7639	8072	8506	8939	9373	9806	0239	0673	5 217
											6 260
10021	0009 1106	1540	1973	2406	2840	3273	3706	4140	4573	5006	7 304
22		5440	5873	6307	6740	7173	7606	8040	8473	8906	9340	8 347
23		9773	0206	0640	1073	1506	1939	2373	2806	3239	3673	9 391
24	0010 4106	4539	4972	5406	5839	6272	6705	7138	7572	8005	
25		8438	8871	9305	9738	0171	0604	1037	1471	1904	2337	
26	0011 2770	3203	3636	4070	4503	4936	5369	5802	6235	6668	
27		7101	7535	7968	8401	8834	9267	9700	0133	0566	0999	433
28	0012 1433	1866	2299	2732	3165	3598	4031	4464	4897	5330	1 43
29		5763	6196	6629	7062	7495	7928	8361	8794	9227	9660	2 87
30	0013 0093	0526	0959	1392	1825	2258	2691	3124	3557	3990	3 130
											4 173
10031		4423	4856	5289	5722	6155	6588	7021	7454	7887	8319	5 217
32		8752	9185	9618	0051	0484	0917	1350	1783	2215	2648	6 260
33	0014 3081	3514	3947	4380	4813	5246	5678	6111	6544	6977	7 303
34		7410	7842	8275	8708	9141	9574	0007	0439	0872	1305	8 346
35	0015 1738	2170	2603	3036	3469	3902	4334	4767	5200	5633	9 390
36		6065	6498	6931	7363	7796	8229	8662	9094	9527	9960	
37	0016 0392	0825	1258	1690	2123	2556	2988	3421	3854	4286	
38		4719	5152	5584	6017	6450	6882	7315	7748	8180	8613	
39		9045	9478	9911	0343	0776	1208	1641	2074	2506	2939	432
40	0017 3371	3804	4236	4669	5102	5534	5967	6399	6832	7264	1 43
											2 86
10041		7697	8129	8562	8994	9427	9859	0292	0724	1157	1589	3 130
42	0018 2022	2454	2887	3319	3752	4184	4616	5049	5481	5914	4 173
43		6346	6779	7211	7644	8076	8508	8941	9373	9806	0238	5 216
44	0019 0670	1103	1535	1968	2400	2832	3265	3697	4129	4562	6 259
45		4994	5426	5859	6291	6723	7156	7588	8020	8453	8885	7 302
											8 345
46		9317	9750	0182	0614	1047	1479	1911	2343	2776	3208	9 389
47	0020 3640	4072	4505	4937	5369	5801	6234	6666	7098	7530	
48		7963	8395	8827	9259	9691	0124	0556	0988	1420	1852	
49	0021 2285	2717	3149	3581	4013	4445	4878	5310	5742	6174	
10050		6606	7038	7470	7903	8335	8767	9199	9631	0063	0495	
	0022										

186

10059

No.	0	1	2	3	4	5	6	7	8	9	Diff.
10050	0021 6606	7038	7470	7903	8335	8767	9199	9631	0063	0495	
51	0022 0927	1359	1791	2224	2656	3088	3520	3952	4384	4816	
52		5248	5680	6112	6544	6976	7408	7840	8272	8704	9136	
53		9568	0000	0432	0864	1296	1728	2160	2592	3024	3456	433
54	0023 3888	4320	4752	5184	5616	6048	6480	6912	7344	7776	1 43
55		8207	8639	9071	9503	9935	0367	0799	1231	1663	2095	2 87
56	0024 2526	2958	3390	3822	4254	4686	5118	5549	5981	6413	3 130
57		6845	7277	7709	8140	8572	9004	9436	9868	0300	0731	4 173
58	0025 1163	1595	2027	2458	2890	3322	3754	4186	4617	5049	5 217
59		5481	5913	6344	6776	7208	7639	8071	8503	8935	9366	6 260
60		9798	0230	0661	1093	1525	1957	2388	2820	3252	3683	7 303
											8 346
10061	0026 4115	4547	4978	5410	5842	6273	6705	7136	7568	8000	9 390
62		8431	8863	9295	9726	0158	0589	1021	1453	1884	2316	
63	0027 2747	3179	3610	4042	4474	4905	5337	5768	6200	6631	
64		7063	7494	7926	8357	8789	9220	9652	0083	0515	0946	
65	0028 1378	1809	2241	2672	3104	3535	3967	4398	4830	5261	432
66		5693	6124	6555	6987	7418	7850	8281	8713	9144	9575	1 43
67	0029 0007	0438	0870	1301	1732	2164	2595	3027	3458	3889	2 86
68		4321	4752	5183	5615	6046	6477	6909	7340	7771	8203	3 130
69		8634	9065	9497	9928	0359	0791	1222	1653	2084	2516	4 173
70	0030 2947	3378	3810	4241	4672	5103	5535	5966	6397	6828	5 216
											6 259
10071		7260	7691	8122	8553	8984	9416	9847	0278	0709	1141	7 302
72	0031 1572	2003	2434	2865	3296	3728	4159	4590	5021	5452	8 346
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74	0032 0195	0626	1057	1488	1919	2350	2781	3212	3643	4074	
75		4505	4937	5368	5799	6230	6661	7092	7523	7954	8385	
76		8816	9247	9678	0109	0540	0971	1402	1833	2264	2695	
77	0033 3126	3557	3988	4419	4850	5281	5712	6143	6574	7004	431
78		7435	7866	8297	8728	9159	9590	0021	0452	0883	1314	1 43
79	0034 1745	2175	2606	3037	3468	3899	4330	4761	5192	5622	2 86
80		6053	6484	6915	7346	7777	8207	8638	9069	9500	9931	3 129
											4 172
10081	0035 0361	0792	1223	1654	2085	2515	2946	3377	3808	4239	5 216
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84	0036 3284	3714	4145	4576	5006	5437	5868	6298	6729	7160	8 345
85		7590	8021	8452	8882	9313	9743	0174	0605	1035	1466	9 388
86	0037 1896	2327	2758	3188	3619	4049	4480	4910	5341	5772	
87		6202	6633	7063	7494	7924	8355	8785	9216	9646	0077	
88	0038 0507	0938	1368	1799	2229	2660	3090	3521	3951	4382	
89		4812	5243	5673	6104	6534	6964	7395	7825	8256	8686	430
90		9117	9547	9977	0408	0838	1269	1699	2129	2560	2990	1 43
											2 86
10091	0039 3421	3851	4281	4712	5142	5572	6003	6433	6864	7294	3 129
92		7724	8155	8585	9015	9445	9876	0306	0736	1167	1597	4 172
93	0040 2027	2458	2888	3318	3748	4179	4609	5039	5470	5900	5 215
94		6330	6760	7191	7621	8051	8481	8911	9342	9772	0202	6 258
95	0041 0632	1063	1493	1923	2353	2783	3213	3644	4074	4504	7 301
											8 344
96		4934	5364	5795	6225	6655	7085	7515	7945	8375	8806	9 387
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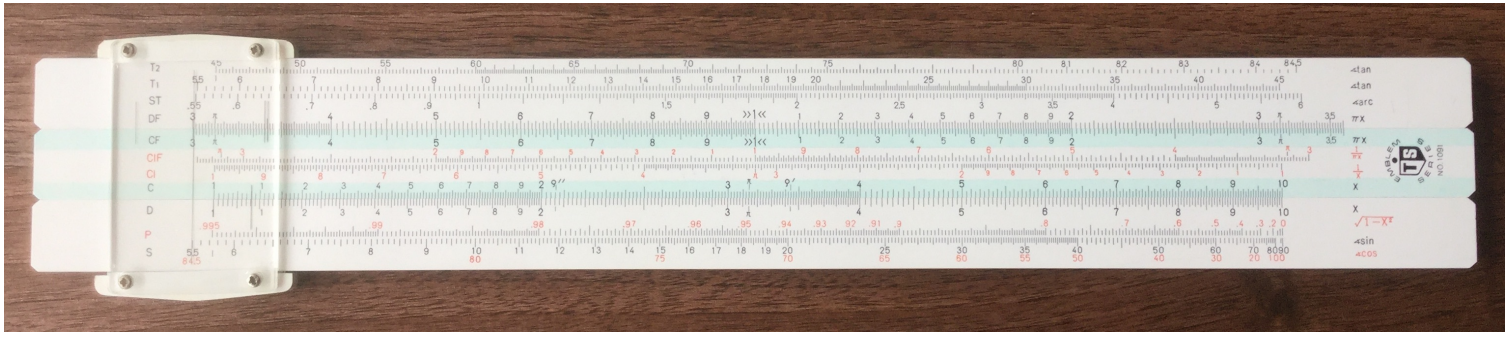
98	0042 3537	3967	4397	4827	5257	5687	6117	6547	6977	7407	
99		7837	8267	8697	9127	9557	9987	0417	0847	1277	1707	
10100	0043 2137	2567	2997	3427	3857	4287	4717	5147	5577	6007	

187

- caught on very quickly
- a calculating aid (until the 1980s)
- logarithms rapidly came to have other interpretations (as you know, and as we shall see)



Significance as a historical source

- Roles of translation in mathematics
- Concept of authorship in the 16th century
- Use of diagrams in mathematical texts
- Importance of informal/social communication, alongside published texts

BO1 History of Mathematics
Lecture III
Analytic geometry and the beginnings of calculus, part 1

MT22 Week 2

Summary

- Brief overview of the 17th century
- A cautionary tale
- Development of notation
- Use of algebra in geometry
- The beginnings of calculus

The 17th century

The main mathematical innovations of the 17th century:

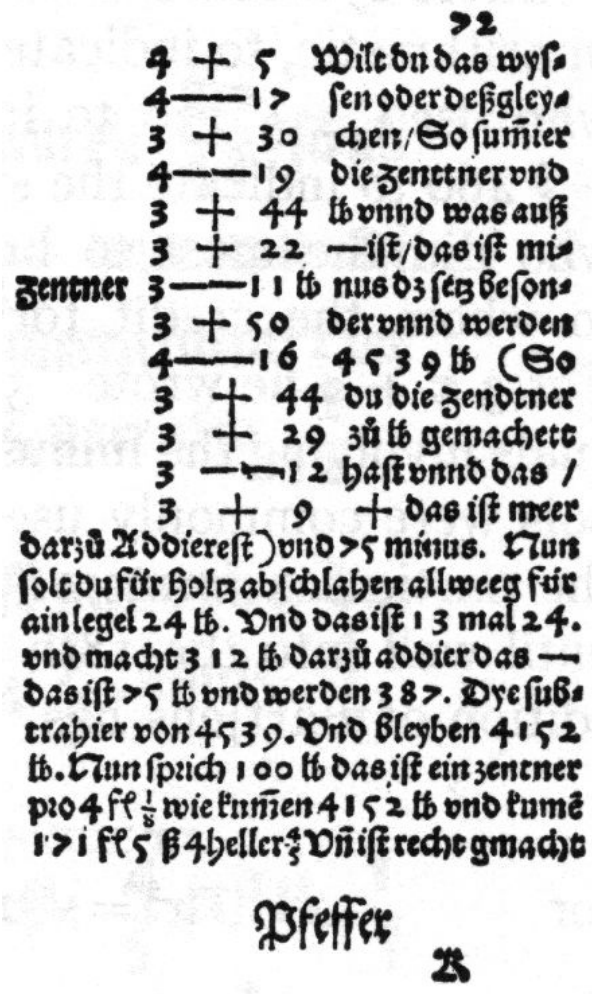
- symbolic notation
- analytic (algebraic) geometry
- calculus
- infinite series [to be treated in later lectures]
- mathematics of the physical world [to be treated in later lectures]

Symbolic notation

Symbolic notation makes mathematics easier

- to read
- to write
- to communicate (though perhaps not orally)
- to think about — and thus stimulates mathematical advances?
- BUT it took a long time to develop
- why did it develop when it did?

Early European notation (abbreviation)



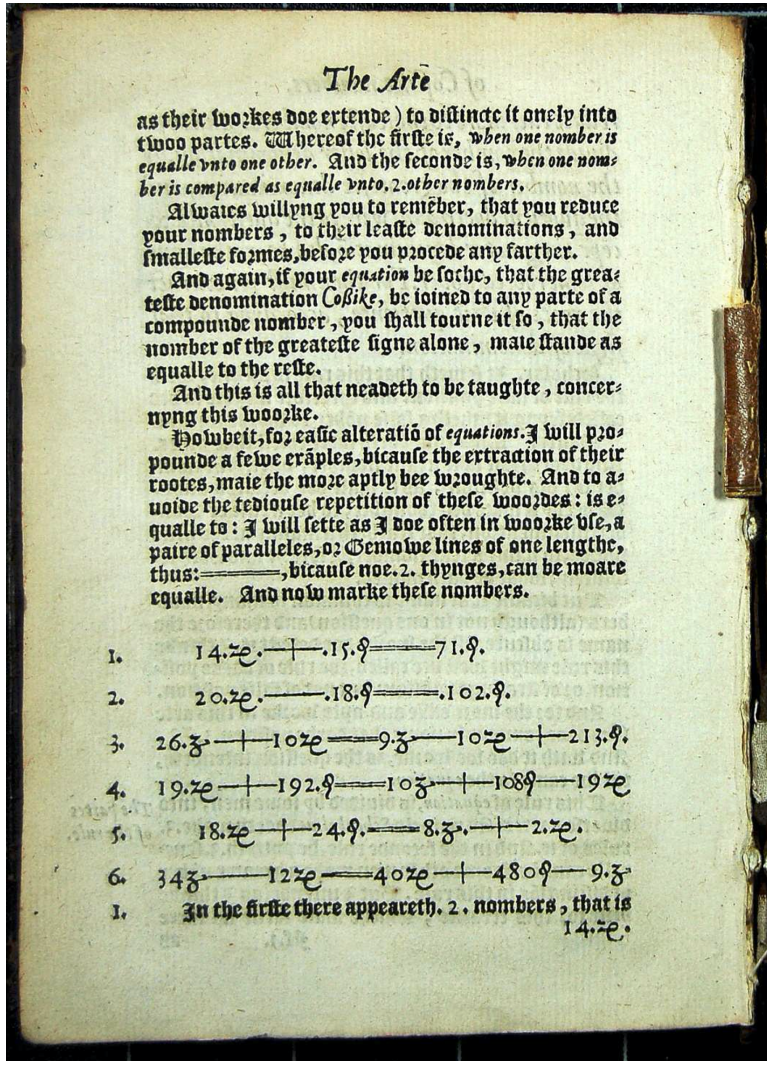
>2

4 + 5 Wilt du das wyſ-
4——1> ſen oder deßgley-
3 + 30 chen/So ſumier
4——19 die Zenttner vnd
3 + 44 ℔ vnnd was auß
3 + 22 —iſt/das iſt mi-
Zentner 3——11 ℔ nus dz ſetz beſon-
3 + 50 der vnnd werden
4——16 4539 ℔ (So
3 + 44 du die Zendtner
3 + 29 zů ℔ gemachett
3 —+—12 haſt vnnd das /
3 + 9 + das iſt meer
darzů Addiereſt) vnd >5 minus. Nun
ſolt du für Holtz abſchlahen allweeg für
ain legel 24 ℔. Vnd das iſt 13 mal 24.
vnd macht 312 ℔ darzů addier das —
das iſt >5 ℔ vnd werden 387. Dye ſub-
trahier von 4539. Vnd bleyben 4152
℔. Nun ſprich 100 ℔ das iſt ein zentner
pro 4 ff½ wie kmmen 4152 ℔ vnd kumē
171 ff 5 ß 4 heller⅔ Vñ iſt recht gmacht

Pfeffer

A

Johannes Widman, *Behende und hüpsche Rechenung auff allen Kauffmanschafft* (1489)



The Arte

as their woꝛkes doe extende) to diſtincte it onely into twoo partes. Whereof the firſte is, *when one nomber is equalle vnto one other.* And the ſeconde is, *when one nomber is compared as equalle vnto, 2. other nombers.*

Alwaies willyng you to remēber, that you reduce your nombers, to their leaſte denominations, and ſmalleſte foꝛmes, beſoꝛe you pꝛocede any farther.

And again, if your *equation* be ſoche, that the greateſte denomination *Coſike*, be ioined to any parte of a compounde nomber, you ſhall tourne it ſo, that the nomber of the greateſte ſigne alone, maie ſtande as equalle to the reſte.

And this is all that neadeth to be taughte, concernyng this woꝛke.

Howbeit, foꝛ eaſie alteratiō of *equations*. I will pꝛopounde a fewe exāples, bicauſe the extraction of their rootes, maie the moꝛe aptly bee wꝛoughte. And to auoide the tediouſe repetition of theſe woꝛdes: is equalle to: I will ſette as I doe often in woꝛke vſe, a paire of paralleles, oꝛ Gemowe lines of one lengthe, thus: ═══, bicauſe noe. 2. thynges, can be moare equalle. And now marke theſe nombers.

1. 14.ze. —+— .15.ꝙ ═══ 71.ꝙ.
2. 20.ze. ——— .18.ꝙ ═══ .102.ꝙ.
3. 26.ʒ —+— 10ze ═══ 9.ʒ —— 10ze —+— 213.ꝙ.
4. 19.ze —+— 192.ꝙ ═══ 10ʒ —+— 108ꝙ —— 19ze
5. 18.ze —+— 24.ꝙ. ═══ 8.ʒ. —+— 2.ze.
6. 34ʒ ——— 12ze ═══ 40ze —+— 480ꝙ —— 9.ʒ
1. In the firſte there appeareth. 2. nombers, that is 14.ze.

Robert Recorde, *The Whetstone of Witte* (1557)

The communication of mathematics

Initially entirely verbal — but usually using a set form of words

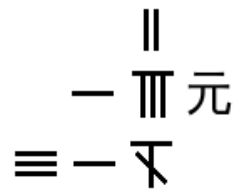
Scribal abbreviations often used

- e.g., later editors of Diophantus (3rd-century Egypt) used ς as an abbreviation for an unknown quantity
- e.g., Bhāskara II (12th-century India) used the initial letters of *yāvattāvat* (*unknown*) and *rūpa* (*unit*) as shorthand: '*yā* 1 *rū* 1' denoted '$x + 1$'

But these were not symbols that could be manipulated algebraically

Arrangement of signs on the page could carry information

- e.g., *tiān yuán shù* 天元術 (13th-century China):



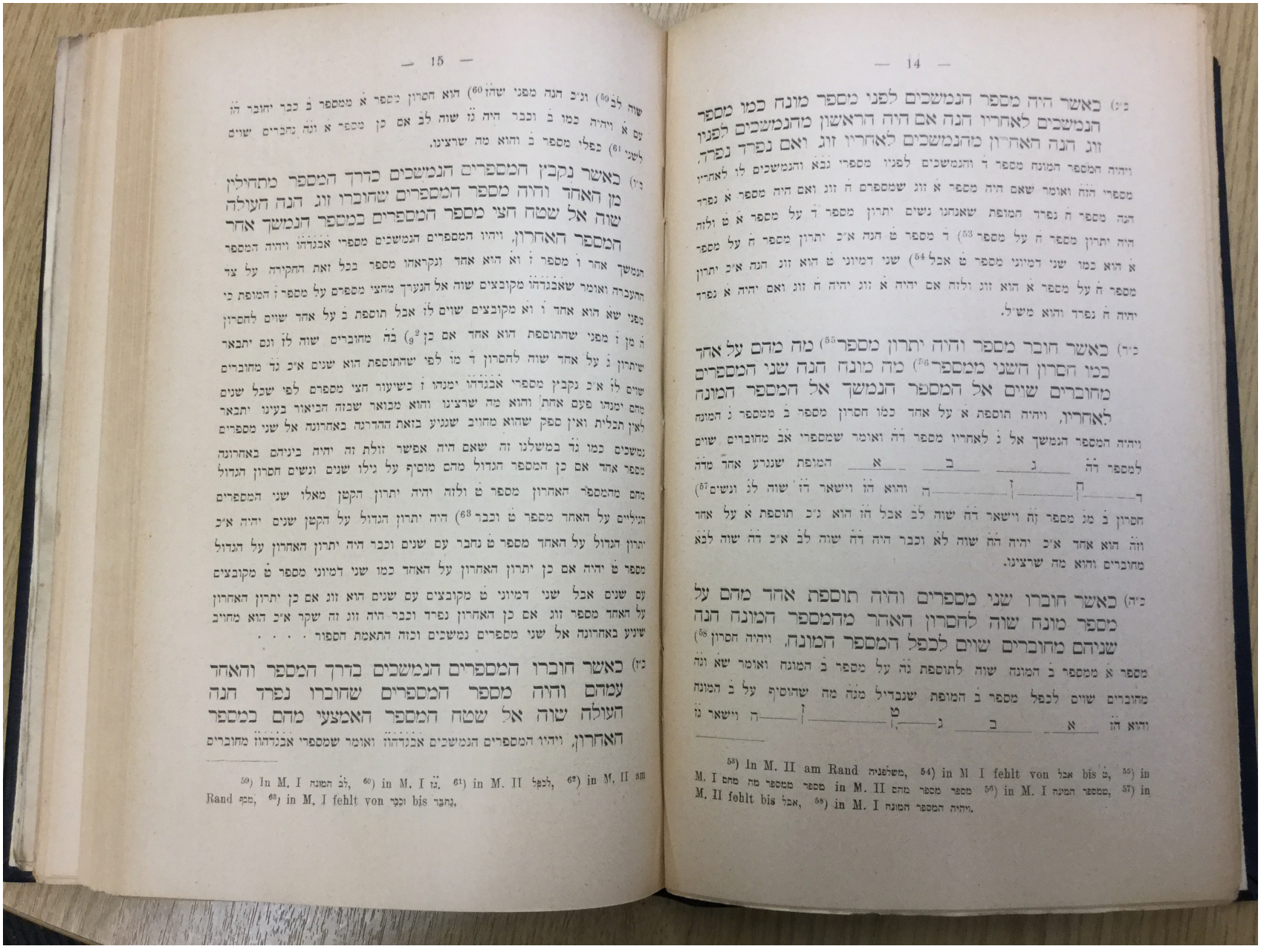
$(2x^2 + 18x - 316)$

Algebraic symbolism of the form that we use came later

A cautionary tale: Levi Ben Gerson and sums of integers

Levi Ben Gerson (Gersonides), *Ma'aseh Hoshev* (*The Work of the Calculator*), 1321 [picture is of a version printed in Venice in 1716]

A cautionary tale: Levi Ben Gerson and sums of integers



— 14 —

כ״ו) כאשר היה מספר הנמשכים לפני מספר מונח כמו מספר הנמשכים לאחריו הנה אם היה הראשון מהנמשכים לפני זוג הנה האחרון מהנמשכים לאחריו זוג ואם נפרד נפרד, ויהיה המספר המונח מספר ד והנמשכים לפניו מספרי גבא והנמשכים לו לאחריו מספרי הה״ ואומר שאם היה מספר א זוג שמספרם ה זוג ואם היה מספר א נפרד הנה מספר ה נפרד המופת שאנחנו נשים יתרון מספר ד על מספר א ט ולזה היה יתרון מספר ה על מספר [53] ד מספר ט הנה א״כ יתרון מספר ה על מספר א הוא כמו שני דמיוני מספר ט אבל [54]) שני דמיוני ט הוא זוג הנה א״כ יתרון מספר ה על מספר א הוא זוג ולזה אם יהיה א זוג יהיה ה זוג ואם יהיה א נפרד יהיה ה נפרד והוא מש״ל.

כ״ז) כאשר חובר מספר והיה יתרון מספר [55]) מה מהם על אחד כמו חסרון השני ממספר [56]) מה מונח הנה שני המספרים מחוברים שוים אל המספר הנמשך אל המספר המונח לאחריו, ויהיה תוספת א על אחד כמו חסרון מספר ב ממספר ג המונח ויהיה המספר הנמשך אל ג לאחריו מספר הה״ ואומר שמספרי אב מחוברים שוים

למספר הה״	ג	ב	א	המופת שננרע אחד מהה״
ד	ח	ז	ה והוא הו וישאר הז שוה לג ונשים [57])	

חסרון ב מן מספר זה וישאר הה״ שוה לב אבל הו הוא ג״כ תוספת א על אחד וזה הוא אחד א״כ יהיה הה״ שוה לא וכבר היה הה״ שוה לב א״כ הה״ שוה לבא מחוברים והוא מה שרצינו.

כ״ח) כאשר חוברו שני מספרים והיה תוספת אחד מהם על מספר מונח שוה לחסרון האחר מהמספר המונח הנה שניהם מחוברים שוים לכפל המספר המונח, ויהיה חסרון [58]) ממספר א ממספר ב המונח שוה לתוספת גה על מספר ב המונח ואומר שא גה מחוברים שוים לכפל מספר ב המופת שנבדיל מגה מה שהוסף על ב המונח והוא הו

	א	ב	ג	ט	ה וישאר גז

[58]) In M. II am Rand משלפניה, [54]) in M I fehlt von אבל bis ב, [55]) in M. I מספר ממספר מה מחם in M. II מספר מספר [56]) in M. I המונח ממספר, [57]) in M. II fehlt bis אבל, [58]) in M. I יהיה המספר המונח.

— 15 —

שוה לב [59]) וג״כ הנה מפני שהו [60]) הוא חסרון מספר א ממספר ב כבר יחובר הו עם א ויהיה כמו ב וכבר היה גז שוה לב אם כן מספר א וגה נחברים שוים לשני [61]) בפלי מספר ב והוא מה שרצינו.

כ״ט) כאשר נקבץ המספרים הנמשכים כדרך המספר מתחילין מן האחד והיה מספר המספרים שחוברו זוג הנה העולה שוה אל שטה חצי מספר המספרים במספר הנמשך אחר המספר האחרון, ויהיו המספרים הנמשכים מספרי אבגדהו ויהיה המספר הנמשך אחר ו מספר ז ואז הוא אחד ונקראהו מספר בכל זאת החקירה על צד ההעברה ואומר שאבגדהו מקובצים שוה אל הערך מחצי מספרים על מספר ז המופת כי מפני שא הוא אחד ו וא מקובצים שוים לו אבל תוספת ב על אחד שוים לחסרון ה מן ז מפני שהתוספת הוא אחד אם כן [62]) בה מחוברים שוה לו וגם יתבאר שיתרון ג על אחד שוה לחסרון ד מו לפי שהתוספת הוא שנים א״כ גד מחוברים שוים לו א״כ נקבץ מספרי אבגדהו ימנחו ז כשיעור הצי מספרם לפי שכל שנים מהם ימנחו פעם אחת והוא מה שרצינו והוא מבואר שבזה הביאור בעינו יתבאר לאין תכלית ואין ספק שהוא מחויב שנגיע בזאת ההדרגה באחרונה אל שני מספרים נמשכים כמו גד במשלנו זה שאם היה אפשר זולת זה יהיה ביניהם באחרונה מספר אחד אם כן המספר הנדול מהם מוסיף על גילו שנים ונשים חסרון הנדול מהם מהמספר האחרון מספר ט ולזה יהיה יתרון הקטן מאלו שני המספרים הנוליים על האחד מספר ט וכבר [63]) היה יתרון הנדול על הקטן שנים יהיה א״כ יתרון הנדול על האחד מספר ט נחבר עם שנים וכבר היה יתרון האחרון על הנדול מספר ט יהיה אם כן יתרון האחרון על האחד כמו שני דמיוני מספר ט מקובצים עם שנים אבל שני דמיוני ט מקובצים עם שנים הוא זוג אם כן יתרון האחרון על האחד מספר זוג אם כן האחרון נפרד וכבר היה זוג זה שקר א״כ הוא מחויב שיגיע באחרונה אל שני מספרים נמשכים וכזה התאמת הספור . . . .

ל׳) כאשר חוברו המספרים הנמשכים בדרך המספר והאחד עמהם והיה מספר המספרים שחוברו נפרד הנה העולה שוה אל שטה המספר האמצעי מהם כמספר האחרון, ויהיו המספרים הנמשכים אבגדהוז ואומר שמספרי אבגדהוז מחוברים

[59]) In M. I לב המונה, [60]) in M. I גז, [61]) in M. II לבגל, [62]) in M. II am Rand מגה, [63]) in M. I fehlt von וכבר bis נחבר,

A cautionary tale: Levi Ben Gerson and sums of integers

Book I, Proposition 26:

> *If we add all consecutive numbers from one to any given number and the given number is even, then the addition equals the product of half the number of numbers that are added up times the number that follows the given even number.*

Book I, Proposition 27:

> *If we add all consecutive numbers from one to any given number and the given number is odd, then the addition equals the product of the number at half way times the last number that is added.*

(Translations from Hebrew by Leo Corry.)

A cautionary tale: Levi Ben Gerson and sums of integers

Converting these into modern notation, we get:

Book I, Proposition 26:

If n is an even number, then $1 + 2 + 3 + \cdots + n = \frac{n}{2}(n + 1)$.

Book I, Proposition 27:

If n is an odd number, then $1 + 2 + 3 + \cdots + n = \frac{n+1}{2} n$.

The formulae are clearly the same, so why are these treated as separate propositions? The answer lies in the proofs, which, like the results themselves, are entirely verbal.

A cautionary tale: Levi Ben Gerson and sums of integers

A fundamental problem here lies in the difficulty of expressing the notion of 'any given number' (our 'n').

A commonly adopted solution was to outline the proof for a specific example, on the understanding that the reader should then be able to adapt the method to any other instance.

Ben Gerson's proof of Proposition 26 takes this approach, and is based on the idea of forming pairs of numbers with equal sums.*

*You might have heard a story about the young Gauss doing the same thing.

A cautionary tale: Levi Ben Gerson and sums of integers

Proof of Proposition 26:

Take the example of 6. If we add 1 and 6, we get 7 ('the number that follows the given even number'). Notice that 2 is obtained from 1 by adding 1, and that 5 is obtained from 6 by subtracting 1, so 2 added to 5 is the same as 1 added to 6, namely 7. The only remaining pair is 3 and 4, which also add to give 7. The number of pairs is half the given even number, hence the total sum is half the number of numbers that are added up times the number that follows the given even number.

This proof is clearly not valid when the given number is odd, since Ben Gerson would have been required to halve it — but he was working only with (positive) integers

A cautionary tale: Levi Ben Gerson and sums of integers

Proposition 27 therefore needs a separate proof, which similarly does not apply when the given number is even (see Leo Corry, *A brief history of numbers*, OUP, 2015, p. 119)

As Corry notes:

> *For Gersonides, the two cases were really different, and there was no way he could realize that the two situations . . . were one and the same as they are for us.*

Moral: take care when converting historical mathematics into modern terms!

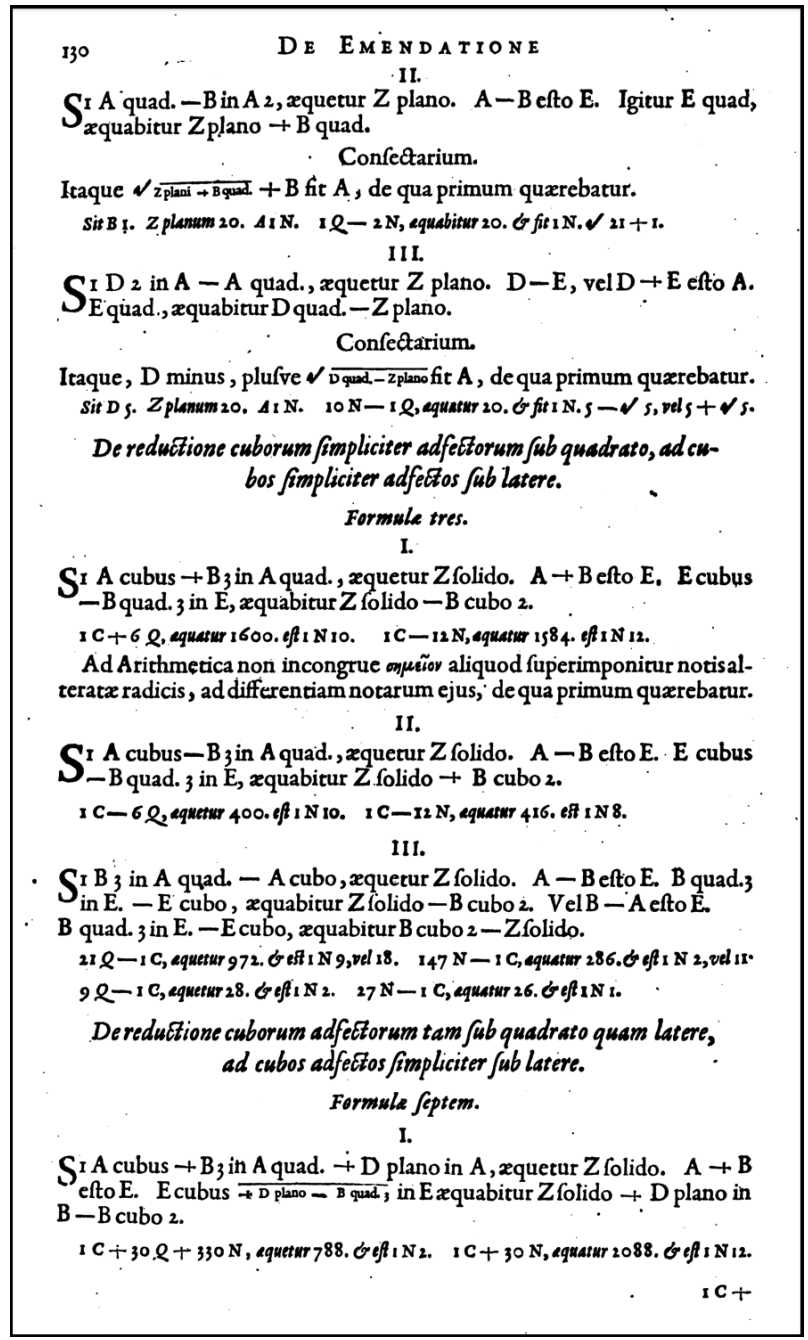
Notation: compare Cardano (*Ars magna*, 1545)...

Having raised a third part of the number of things to a cube, to which you add the square of half the number in the equation and take the root of the total, consider the square [root], which you will take twice; and to one of them you add half of the same, and you will have the binome with its apotome, whence taking the cube root of the apotome from the cube root of its binome, the difference that comes from this, is the value of the thing.

(*Mathematics emerging*, p. 327)

... with Viète (c. 1590)...

François Viète
(Francisci Vieta)
Opera mathematica
1646, p. 130



130 DE EMENDATIONE

II.

SI A quad. — B in A 2, æquetur Z plano. A — B esto E. Igitur E quad, æquabitur Z plano + B quad.

Consectarium.

Itaque $\sqrt{Z\text{ plani} + B\text{ quad.}}$ + B fit A, de qua primum quærebatur.

Sit B 1. *Z planum* 20. *A* 1 N. 1 Q — 2 N, *æquabitur* 20. *& fit* 1 N. $\sqrt{}$ 21 + 1.

III.

SI D 2 in A — A quad., æquetur Z plano. D — E, vel D + E esto A. E quad., æquabitur D quad. — Z plano.

Consectarium.

Itaque, D minus, pluſve $\sqrt{D\text{ quad.} - Z\text{ plano}}$ fit A, de qua primum quærebatur.

Sit D 5. *Z planum* 20. *A* 1 N. 10 N — 1 Q, *æquatur* 20. *& fit* 1 N. 5 — $\sqrt{}$ 5, *vel* 5 + $\sqrt{}$ 5.

De reductione cuborum ſimpliciter adfectorum ſub quadrato, ad cubos ſimpliciter adfectos ſub latere.

Formulæ tres.

I.

SI A cubus + B 3 in A quad., æquetur Z ſolido. A + B esto E. E cubus — B quad. 3 in E, æquabitur Z ſolido — B cubo 2.

1 C + 6 Q, *æquatur* 1600. *eſt* 1 N 10. 1 C — 12 N, *æquatur* 1584. *eſt* 1 N 12.

Ad Arithmetica non incongrue σημεῖον aliquod ſuperimponitur notis alteratæ radicis, ad differentiam notarum ejus, de qua primum quærebatur.

II.

SI A cubus — B 3 in A quad., æquetur Z ſolido. A — B esto E. E cubus — B quad. 3 in E, æquabitur Z ſolido + B cubo 2.

1 C — 6 Q, *æquetur* 400. *eſt* 1 N 10. 1 C — 12 N, *æquatur* 416. *eſt* 1 N 8.

III.

SI B 3 in A quad. — A cubo, æquetur Z ſolido. A — B esto E. B quad. 3 in E. — E cubo, æquabitur Z ſolido — B cubo 2. Vel B — A esto E. B quad. 3 in E. — E cubo, æquabitur B cubo 2 — Z ſolido.

21 Q — 1 C, *æquetur* 972. *& eſt* 1 N 9, *vel* 18. 147 N — 1 C, *æquatur* 286. *& eſt* 1 N 2, *vel* 11.

9 Q — 1 C, *æquetur* 28. *& eſt* 1 N 2. 27 N — 1 C, *æquatur* 26. *& eſt* 1 N 1.

De reductione cuborum adfectorum tam ſub quadrato quam latere, ad cubos adfectos ſimpliciter ſub latere.

Formulæ ſeptem.

I.

SI A cubus + B 3 in A quad. + D plano in A, æquetur Z ſolido. A + B esto E. E cubus $\overline{+ D\text{ plano} - B\text{ quad. 3}}$ in E æquabitur Z ſolido + D plano in B — B cubo 2.

1 C + 30 Q + 330 N, *æquetur* 788. *& eſt* 1 N 2. 1 C + 30 N, *æquatur* 2088. *& eſt* 1 N 12.

1 C +

... with Viète (c. 1590)...

II.

SI A quad. — B in A 2, æquetur Z plano. A — B esto E. Igitur E quad, æquabitur Z plano + B quad.

Consectarium.

Itaque $\checkmark \overline{\text{Z plani} + \text{B quad.}} + \text{B}$ fit A, de qua primum quærebatur.

Sit B 1. Z planum 20. A 1 N. 1 Q — 2 N, equabitur 20. & fit 1 N. ✓ 21 + 1.

III.

SI D 2 in A — A quad., æquetur Z plano. D — E, vel D + E esto A. E quad., æquabitur D quad. — Z plano.

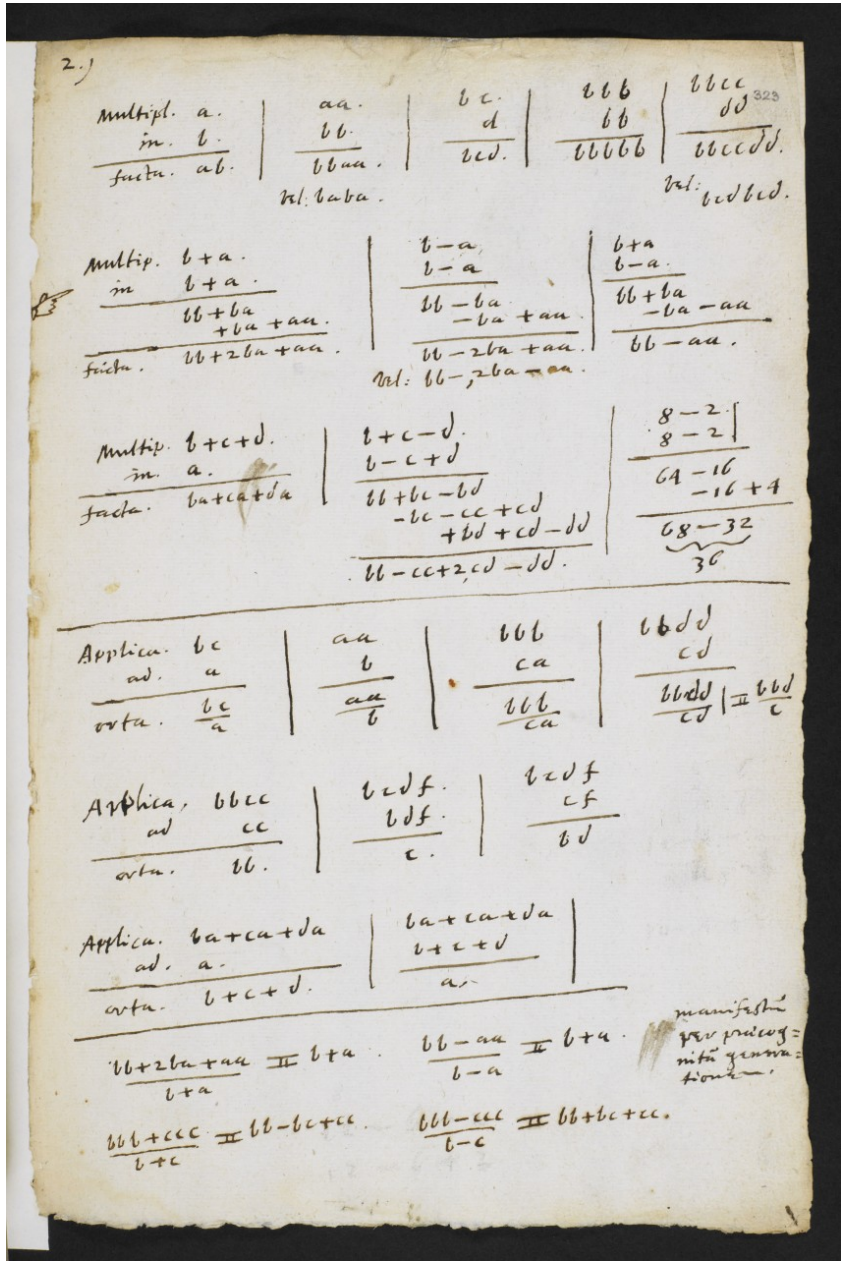
Consectarium.

Itaque, D minus, pluſve $\checkmark \overline{\text{D quad.} - \text{Z plano}}$ fit A, de qua primum quærebatur.

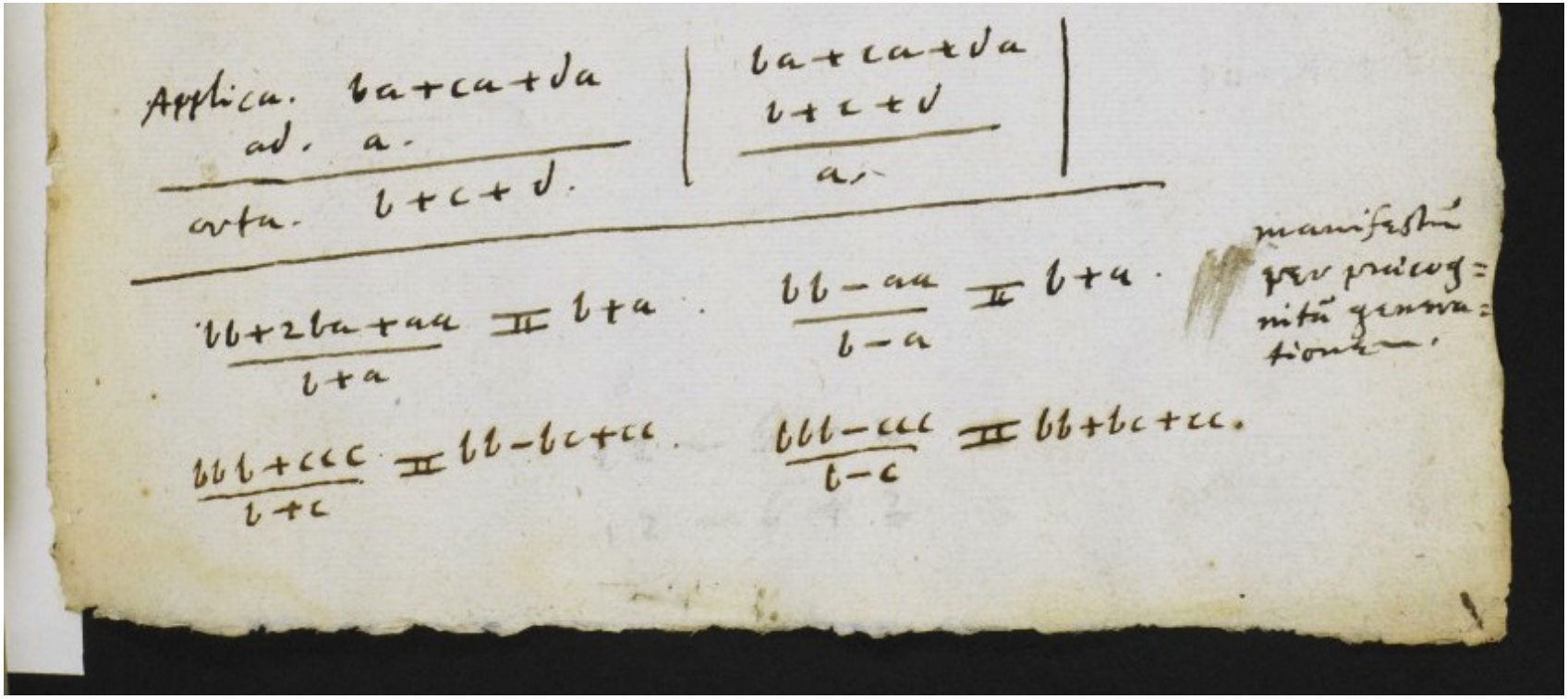
Sit D 5. Z planum 20. A 1 N. 10 N — 1 Q, equatur 20. & fit 1 N. 5 — ✓ 5, vel 5 + ✓ 5.

... and with Harriot (c. 1600)

British Library
Add MS 6784 f. 323
available at
Thomas Harriot Online

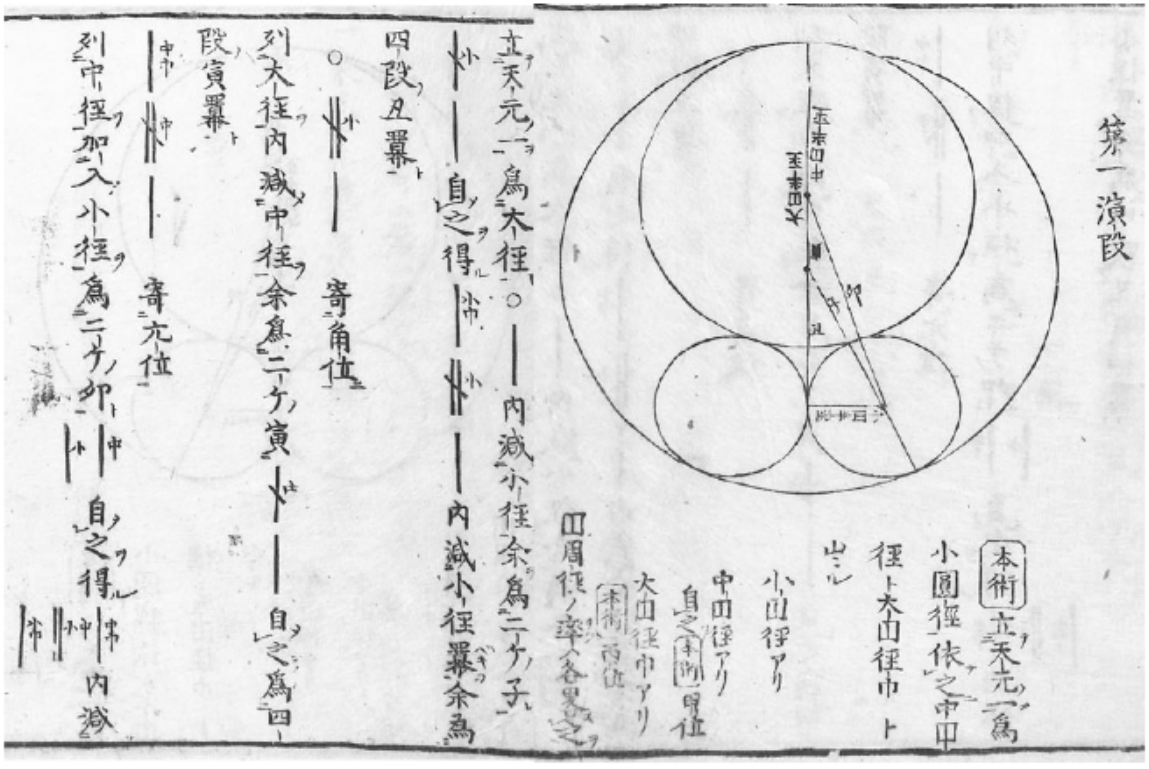


... and with Harriot (c. 1600)



Elsewhere in the world

Seki Takakazu, *Hatsubi Sanpō* 発微算法 (1674), concerning the solution of equations in several variables:



Equations written using the technique of *bōshohō* 傍書法 (‘side-writing’; a.k.a. *tenzan jutsu* 点竄術)

Notation: Viète (Tours, c. 1590)

François Viète (1540–1603, France):

A, E, ... (i.e., vowels) for unknowns

B, C, D, ... (i.e., consonants) for known or given quantities

symbols $+$, $-$

but otherwise verbal descriptions and connections: quadratum (squared), cubus (cubed), aequatur (be equal), ...

Notation: Harriot (London, c. 1600)

Thomas Harriot (1560–1621, England):

a, e, ... for unknowns

b, c, d, ... for known or given quantities

+, −

ab, aa, aaa

and many symbols: =, >, ...

(For another example of Harriot's use of notation, see *Mathematics emerging*, §2.2.1.)

Harriot papers online: http://echo.mpiwg-berlin.mpg.de/content/scientific_revolution/harriot

Notation: Descartes (Netherlands, 1637)

René Descartes (1596–1650, France and Holland):

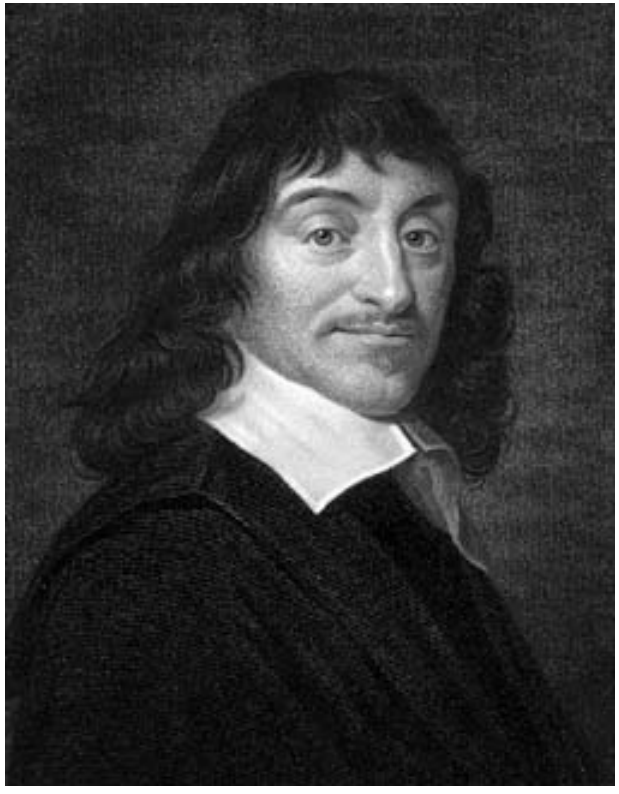
x, y, ... for unknowns

a, b, c, ... for known or given quantities

$+$, $-$

xx, x^3, x^4, ...

Descartes' notation was widely adopted, although his '∞' for equality eventually gave way to '$=$', and his '$\sqrt{C}$' to '$\sqrt[3]{}$'.



Descartes' notation

tirer de cete ſcience. Auſſy que ie n'y remarque rien de ſi difficile, que ceux qui ſeront vn peu verſés en la Geometrie commune, & en l'Algebre, & qui prendront garde a tout ce qui eſt en ce traité, ne puiſſent trouuer.

C'eſt pourquoy ie me contenteray icy de vous auertir, que pourvû qu'en demeſlant ces Equations on ne manque point a ſe ſeruir de toutes les diuiſions, qui ſeront poſſibles, on aura infalliblement les plus ſimples termes, auſquels la queſtion puiſſe eſtre reduite.

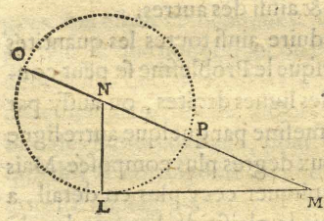
Quels ſont les probleſmes plans

Et que ſi elle peut eſtre reſolue par la Geometrie ordinaire, c'eſt a dire, en ne ſe ſeruant que de lignes droites & circulaires tracées ſur vne ſuperficie plate, lorſque la derniere Equation aura eſté entierement démeſlée, il n'y reſtera tout au plus qu'vn quarré inconnu, eſgal a ce qui ſe produiſt de l'Addition, ou ſouſtraction de ſa racine multipliée par quelque quantité connue, & de quelque autre quantité auſſy connue.

Comment ils ſe reſoluent.

Et lors cete racine, ou ligne inconnue ſe trouue ayſement. Car ſi i'ay par exemple

$$z^2 \propto az + bb$$

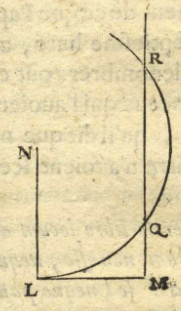


ie fais le triangle rectangle N L M, dont le coſté L M eſt eſgal à b racine quarrée de la quantité connue bb, & l'autre L N eſt $\frac{1}{2}a$, la moitié de l'autre quantité connue, qui eſtoit multipliée par z que ie ſuppoſe eſtre la ligne inconnue. puis prolongeant M N la baze de ce triangle,

angle, iuſques a O, en ſorte qu'N O ſoit eſgale a N L, la toute O M eſt z la ligne cherchée. Et elle s'exprime en cete ſorte

$$z \propto \tfrac{1}{2}a + \sqrt{\tfrac{1}{4}aa + bb}.$$

Que ſi iay $yy \propto -ay + bb$, & qu'y ſoit la quantité qu'il faut trouuer, ie fais le meſme triangle rectangle N L M, & de ſa baze M N i'oſte N P eſgale a N L, & le reſte P M eſt y la racine cherchée. De façon que iay $y \propto -\tfrac{1}{2}a + \sqrt{\tfrac{1}{4}aa + bb}$. Et tout de meſme ſi i'auois $x^4 \propto -ax^2 + b^2$. P M ſeroit x^2. & i'aurois $x \propto \sqrt{-\tfrac{1}{2}a + \sqrt{\tfrac{1}{4}aa + bb}}$: & ainſi des autres.



Enfin ſi i'ay

$$z^2 \propto az - bb:$$

ie fais N L eſgale à $\frac{1}{2}a$, & L M eſgale à b cõme deuãt, puis, au lieu de ioindre les poins M N, ie tire M Q R parallele a L N. & du centre N par L ayant deſcrit vn cercle qui la couppe aux poins Q & R, la ligne cherchée z eſt M Q, oubiẽ M R, car en ce cas elle s'exprime en deux façons, a ſçauoir $z \propto \tfrac{1}{2}a + \sqrt{\tfrac{1}{4}aa - bb}$, & $z \propto \tfrac{1}{2}a - \sqrt{\tfrac{1}{4}aa - bb}$.

Et ſi le cercle, qui ayant ſon centre au point N, paſſe par le point L, ne couppe ny ne touche la ligne droite M Q R, il n'y a aucune racine en l'Equation, de façon qu'on peut aſſurer que la conſtruction du probleſme propoſé eſt impoſſible.

Au

Symbolism established in algebra

Frontispiece to: Johannes Faulhaber, *Ingenieurs-Schul, Anderer Theil*, Ulm, 1633 (on fortification)

See: Volker Remmert, 'Antiquity, nobility, and utility: picturing the Early Modern mathematical sciences', in *The Oxford handbook of the history of mathematics* (Eleanor Robson & Jacqueline Stedall, eds.), OUP, 2009, pp. 537–563

‘Analysis’ vs ‘synthesis’

Viète (and others) sought to ‘restore’ ancient Greek mathematical ideas — in particular, those found in the recently rediscovered *Collection* (or *Synagoge*: $\Sigma\upsilon\nu\alpha\gamma\omega\gamma\acute{o}\acute{\eta}$) of Pappus of Alexandria (4th century AD) [published in Latin by Federico Commandino in 1588]

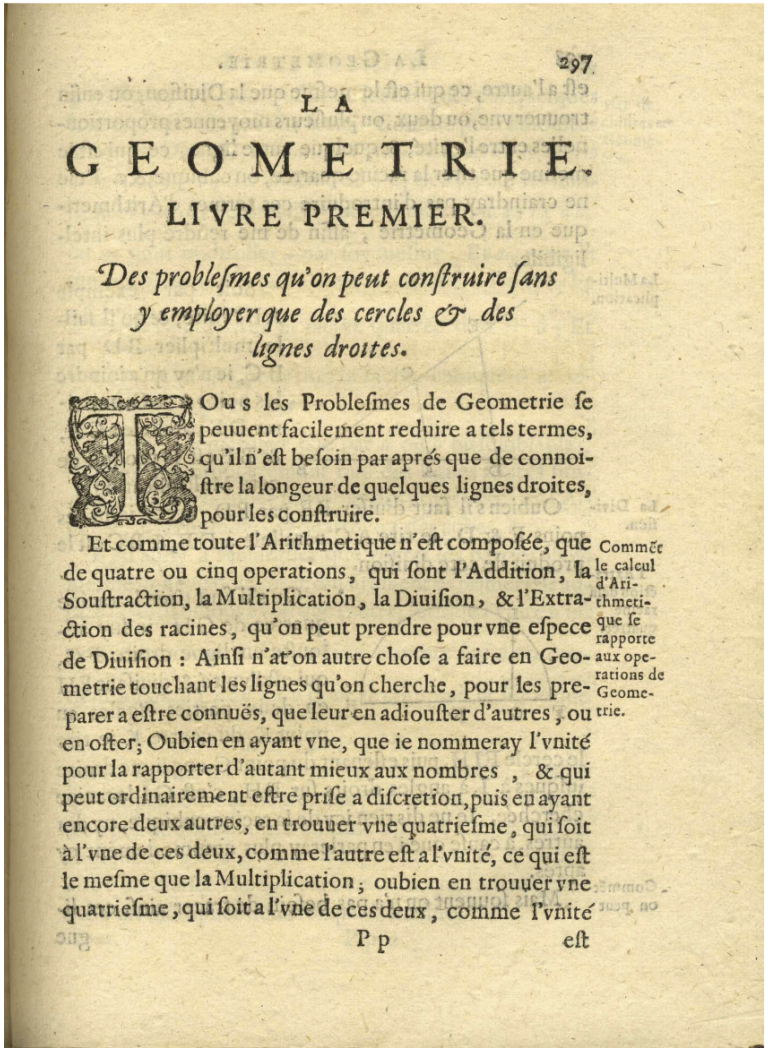
Book VII of Pappus’s *Collection* outlines the methods of analysis and synthesis:

Synthesis: starting from what is known, we make a sequence of deductions until we arrive at what is sought (constructive method, as, e.g., in Euclid’s *Elements*)

Analysis: starting from what is sought, as if it has already been established, we work backwards until we arrive at what is known (method of discovery or problem-solving, preliminary to synthesis)

“Analysis was thus the working tool of the geometer, but it was with synthesis that one could demonstrate things in an indisputable way.” (Niccolò Guicciardini, ‘Analysis and synthesis in Newton’s mathematical work’, *The Cambridge Companion to Newton* (ed. I. Bernard Cohen and George E. Smith), CUP, 2002, pp. 308–328 at p. 308)

Analytic (algebraic) geometry



297

LA

GEOMETRIE.

LIVRE PREMIER.

Des problesmes qu'on peut construire sans y employer que des cercles & des lignes droites.

TOus les Problesmes de Geometrie se peuuent facilement reduire a tels termes, qu'il n'est besoin par aprés que de connoistre la longeur de quelques lignes droites, pour les construire.

Et comme toute l'Arithmetique n'est composée, que de quatre ou cinq operations, qui sont l'Addition, la Soustraction, la Multiplication, la Diuision, & l'Extraction des racines, qu'on peut prendre pour vne espece de Diuision : Ainsi n'at'on autre chose a faire en Geometrie touchant les lignes qu'on cherche, pour les preparer a estre connuës, que leur en adiouster d'autres, ou en oster; Oubien en ayant vne, que ie nommeray l'vnité pour la rapporter d'autant mieux aux nombres, & qui peut ordinairement estre prise a discretion, puis en ayant encore deux autres, en trouuer vne quatriesme, qui soit à l'vne de ces deux, comme l'autre est a l'vnité, ce qui est le mesme que la Multiplication; oubien en trouuer vne quatriesme, qui soit a l'vne de ces deux, comme l'vnité

Comme le calcul d'Arithmetique se rapporte aux operations de Geometrie.

Pp est

La géométrie (1637)

Solution of geometric problems by algebraic methods

Appendix to
Discours de la méthode

"by commencing with objects the simplest and easiest to know, I might ascend by little and little"

Descartes' analytic geometry

We may label lines (line segments) with letters a, b, c, ...

Then $a + b$, $a - b$, ab, a/b, $\sqrt{a}$ may be constructed by ruler and compass.

Descartes' method

- represent all lines by letters
- use the conditions of the problem to form equations
- reduce the equations to a single equation
- solve
- construct the solution geometrically

For examples, see Katz (3rd ed.), §14.2

Algebraic methods in geometry: some objections

Pierre de Fermat (1656, France):

> *I do not know why he has preferred this method with algebraic notation to the older way which is both more convincing and more elegant ...*

Thomas Hobbes (1656, England):

> *... a scab of symbols ...*

The beginnings of calculus: tangent methods

Calculus:

- finding tangents;
- finding areas.

Descartes' method for finding tangents (1637)

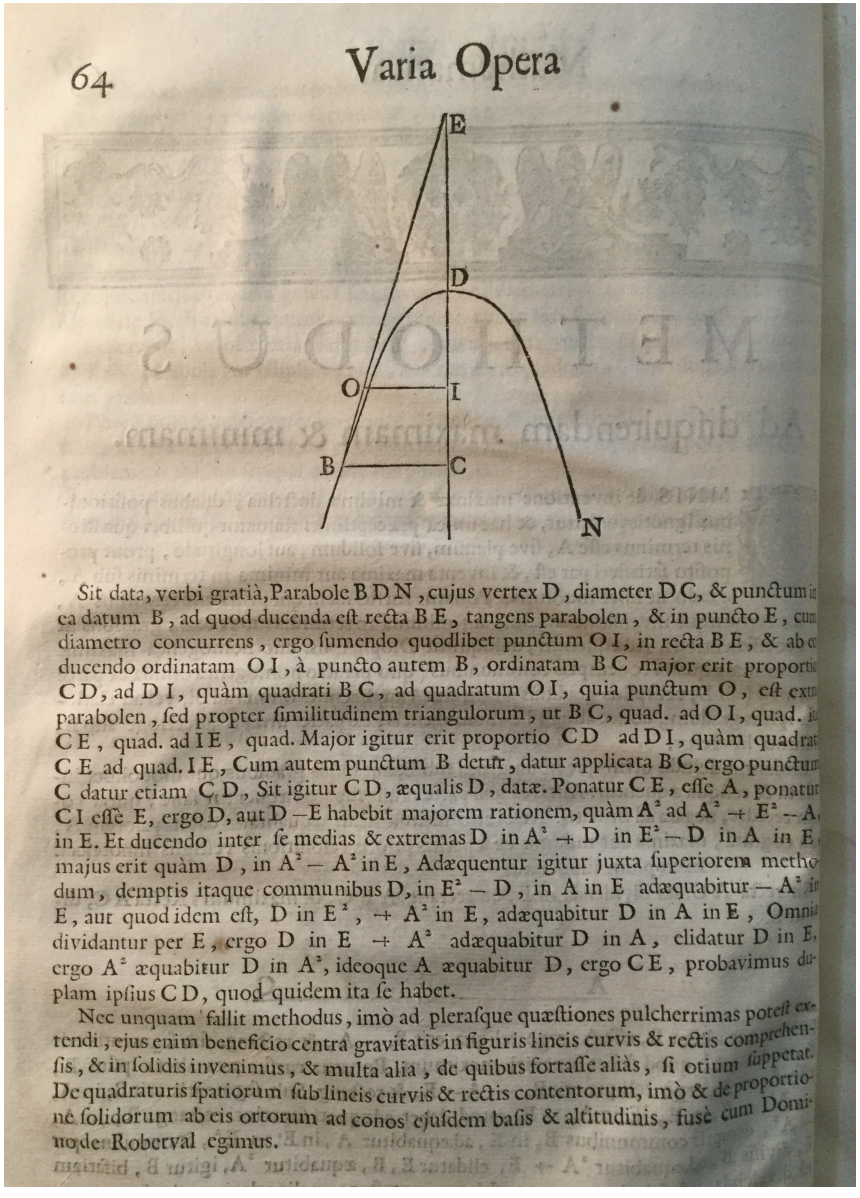
- based on finding a circle that touches the curve at the given point — a tangent to the circle is then a tangent to the curve
- used his algebraic approach geometry to find double roots to equation of intersection
- was in principle a general method — but laborious

Fermat's method for finding tangents

Pierre de Fermat (1601–1665):

- steeped in classical mathematics
- like Descartes, investigated problems of Pappus
- devised a tangent method (1629) quite different from that of Descartes

Fermat's tangent method (1629)



64 Varia Opera

E D O I B C N

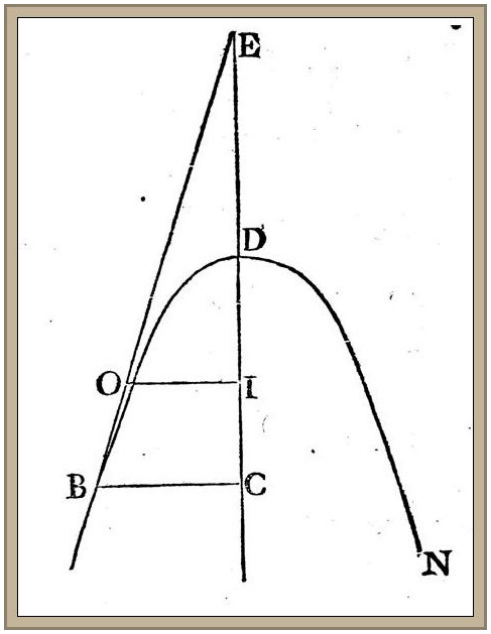
Sit data, verbi gratiâ, Parabole B D N, cujus vertex D, diameter D C, & punctum in ea datum B, ad quod ducenda eſt recta B E, tangens parabolen, & in puncto E, cum diametro concurrens, ergo ſumendo quodlibet punctum O I, in recta B E, & ab o ducendo ordinatam O I, à puncto autem B, ordinatam B C major erit proporti C D, ad D I, quàm quadrati B C, ad quadratum O I, quia punctum O, eſt extr parabolen, ſed propter ſimilitudinem triangulorum, ut B C, quad. ad O I, quad. i C E, quad. ad I E, quad. Major igitur erit proportio C D ad D I, quàm quadrat C E ad quad. I E, Cum autem punctum B detur, datur applicata B C, ergo punctum C datur etiam C D, Sit igitur C D, æqualis D, datæ. Ponatur C E, eſſe A, ponatur C I eſſe E, ergo D, aut D – E habebit majorem rationem, quàm A^2 ad $A^2 + E^2 - A$, in E. Et ducendo inter ſe medias & extremas D in $A^2 + D$ in $E^2 - D$ in A in E, majus erit quàm D, in $A^2 - A^2$ in E, Adæquentur igitur juxta ſuperiorem metho dum, demptis itaque communibus D, in $E^2 - D$, in A in E adæquabitur $- A^2$ in E, aut quod idem eſt, D in E^2, $+ A^2$ in E, adæquabitur D in A in E, Omnia dividantur per E, ergo D in E $+ A^2$ adæquabitur D in A, elidatur D in E, ergo A^2 æquabitur D in A^2, ideoque A æquabitur D, ergo C E, probavimus du plam ipſius C D, quod quidem ita ſe habet.

Nec unquam fallit methodus, imò ad pleraſque quæſtiones pulcherrimas poteſt extendi, ejus enim beneficio centra gravitatis in figuris lineis curvis & rectis comprehenſis, & in ſolidis invenimus, & multa alia, de quibus fortaſſe aliàs, ſi otium ſuppetat. De quadraturis ſpatiorum ſub lineis curvis & rectis contentorum, imò & de proportione ſolidorum ab eis ortorum ad conos ejuſdem baſis & altitudinis, fuſè cum Domino de Roberval egimus.

Worked out c. 1629, but only published posthumously in *Varia opera mathematica*, 1679.

See *Mathematics emerging*, §3.1.1.

Fermat's tangent method (1629)



Choose an arbitrary point B on the parabola.

Suppose that the tangent at B exists, and that it crosses the axis of the parabola at E.

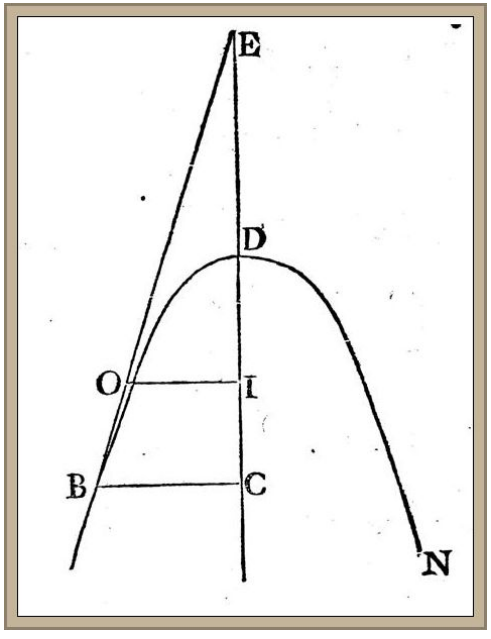
Choose any point O on the line BE.

Draw horizontals OI and BC.

Since O is outside the parabola, we have

$$\frac{CD}{DI} > \frac{(BC)^2}{(OI)^2}.$$

Fermat's tangent method (1629)



Since O is outside the parabola, we have

$$\frac{CD}{DI} > \frac{(BC)^2}{(OI)^2}.$$

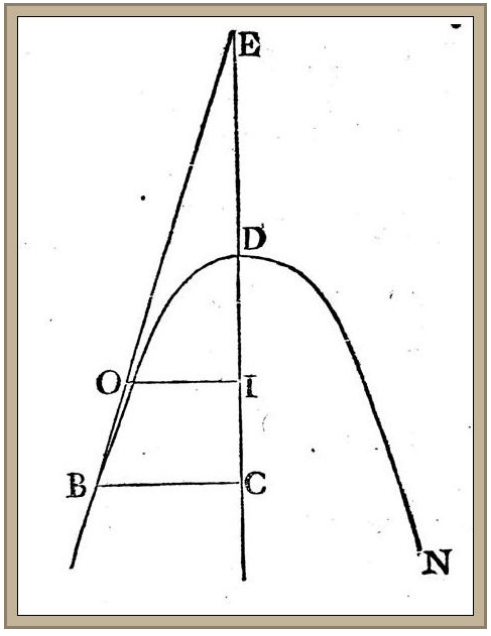
By similarity of triangles,

$$\frac{(BC)^2}{(OI)^2} = \frac{(CE)^2}{(IE)^2}.$$

Therefore

$$\frac{CD}{DI} > \frac{(CE)^2}{(IE)^2}.$$

Fermat's tangent method (1629)



Therefore

$$\frac{CD}{DI} > \frac{(CE)^2}{(IE)^2}.$$

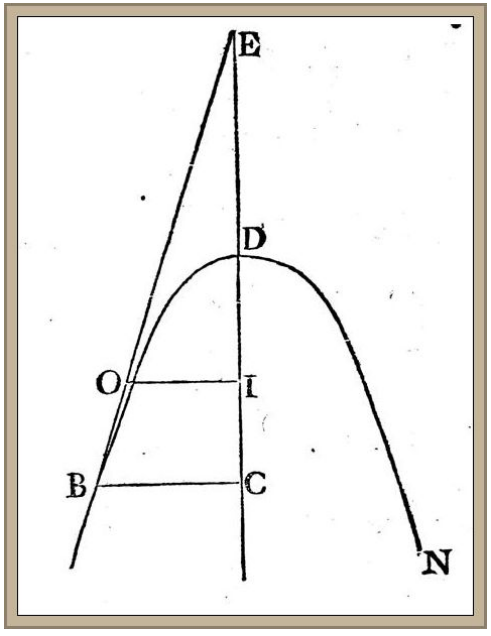
Put $CD = d$, $CE = a$, $CI = e$, so that

$$\frac{d}{d-e} > \frac{a^2}{(a-e)^2}.$$

Now (Fermat says), we obtain equality as e decreases (as OI becomes BC):

$$\frac{d}{d-e} = \frac{a^2}{(a-e)^2}.$$

Fermat's tangent method (1629)



We solve the equality

$$\frac{d}{d-e} = \frac{a^2}{(a-e)^2}.$$

Rearranging gives $de^2 + a^2 e = 2ade.$

Cancel e: $de + a^2 = 2ad.$

Now e will be small, so we can neglect it, leaving us with $a^2 = 2ad.$

Hence $a = 2d.$

Or $CE = 2 \times CD.$

BO1 History of Mathematics
Lecture IV
The beginnings of calculus, part 2: quadrature

MT 2022 Week 2

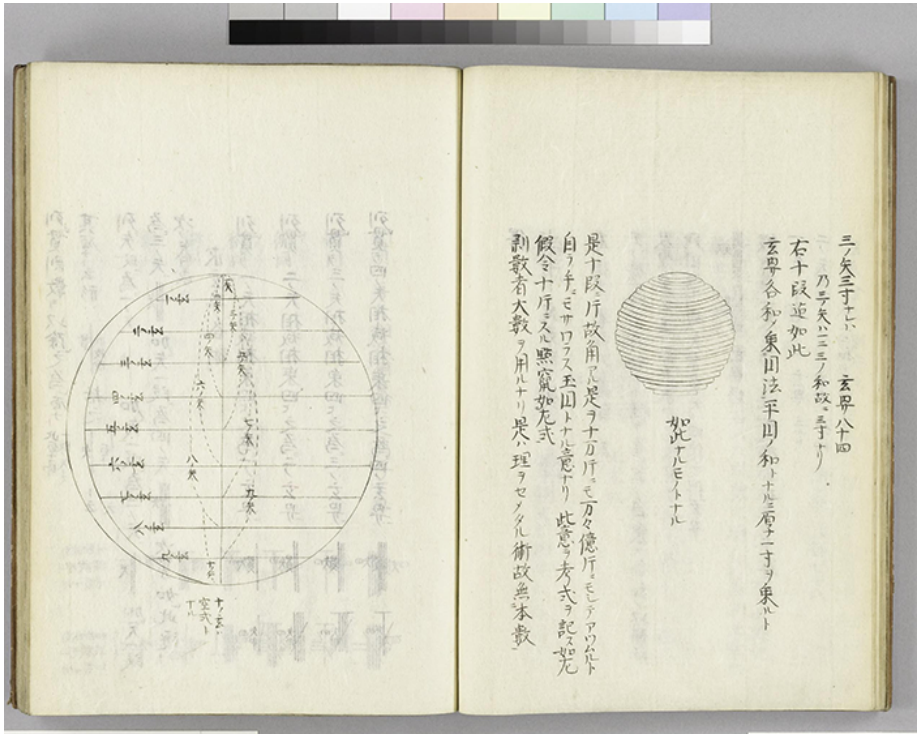
Summary

- *Enri*: a non-Western prelude
- Quadrature (finding areas)
- Indivisibles
- Infinitesimals
- The contributions of Newton & Leibniz

Seki and *enri*

The development of calculus is a largely Western story, but similar ideas did appear elsewhere

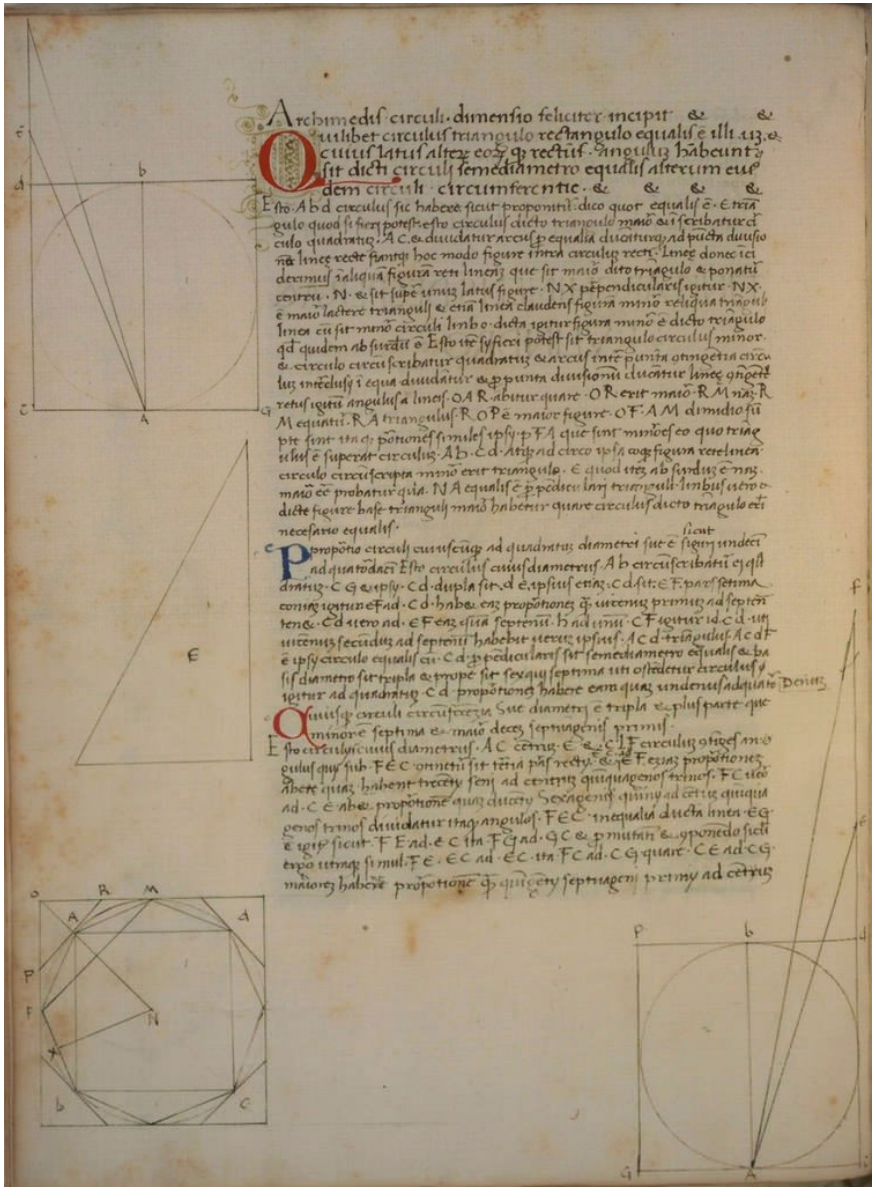
For example, in the late 17th century, Seki Takakazu and his school developed *enri* 円理 ('circle principles'), which concerned the calculation of arc lengths, areas, and volumes



One result of *enri* was the determination of the volume of a sphere via 'slicing'

But *enri* was much narrower in scope than calculus

Archimedes: *Κύκλου μέτρησις* (*Measurement of a circle*)

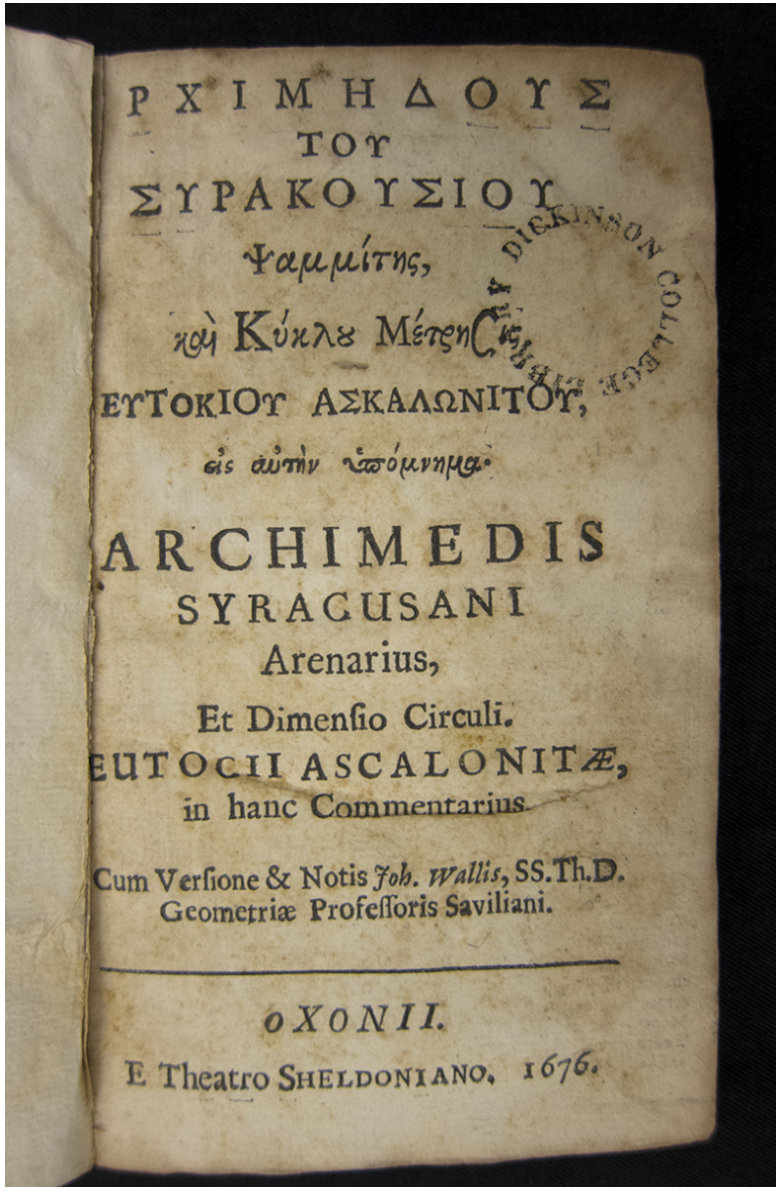

Archimedis circuli dimensio feliciter incipit & & &
Quilibet circulus triangulo rectangulo equalis est illi cuius latus alterum eorum quod rectum angulum habeunt sit dicti circuli semidiametro equalis alterum eius dem circuli circumferentie & & & &
Esto A b d circulus sic habere sicut proponitur dico quod equalis est e triangulo quod si fieri potest esto circulus dicto triangulo maius ut scribatur circulo quadratus A C & diuidantur arcus ipsi equalia ducantur ad puncta diuisionum lineae recte sicque hoc modo figure intra circulum recte lineae donec ici dentur residua figura ren linem que sit maius dicto triangulo & ponatur centru N & sit super unum latus figure N X perpendicularis scilicet N X
e maius habere triangulus & etiam lineae claudens figura minus residua trianguli lineae cui sit minus circuli limbo dicta igitur figura minus est dicto triangulo quidem ab sicut o Esto ut sic sufficiat potest sit triangulo circulus minor
& circulo circumscribatur quadratus & arcus inter puncta omnia circumferentie inclusi equa diuidantur & per puncta diuisionum ducantur lineae quae cet
rotus igitur angulus a lineis O A R habitur quare O R erit maius R M nam R
M equatur R A triangulus R O P est maior figure O F A M dimidio sui
pte sint tria quae portiones similes ipsis per F A quae sint minores eo quo trian
gulus est superat circulus A b C d Atqui ad circo ipsa aggregum recte lineae
circulo circumscripta minus erit triangulo e quod ites ab simplicius est nam
maius est probatur quia N A equalis est perpendiculari trianguli limbus autem o
dicte figure base triangulj minus habentur quare circulus dicto triangulo est
necessario equalis.
Proportio circuli cuiuscunque ad quadratum diametri sue est sicut significatur
ad quatuordecim Esto circulus cuius diameter A b circumscribatur ei quod
dratus C G e ipsi C d dupla sit ad e ipsius enim C d sit E F pars septima
coniunctum igitur E F ad C d habebit eam proportionem quam unum primus ad septem
tenet C d vero ad E F est sua septem habet unus C F igitur id C d un
decimus secundum ad septem habebit vero ipsius A C d triangulus A C d
est ipsi circulo equalis cui C d proximus latus sit semidiametro equalis & ba
sis diametro sit tripla & prope sit sexagesima septima ut ostenditur circulus
igitur ad quadratum C d proportionem habet eam quam undecim ad quatuor
Quilibet circuli circumferentia sue diametri est tripla & plus parte que
minor est septima est autem decem septuagenis primis
Esto circulus cuius diameter A C centrum E & C F circulus autem ad
equalis qui sub F E C omnium sit terna pars recti & E F enim proportionem
habere quam habent tredecim simul ad centum quinquaginta tres. F C uero
ad C E habet proportionem quam ducenti sexaginta quinque ad centum quinqua
ginta tres diuidatur trianguli angulus F E C inequalis ducta linea E G
per tres equas diuidatur trianguli angulus F E C inequalis ducta linea E G
e igitur sicut F E ad E C ita F G ad G C & permutando & componendo sicut
erit una simul F E E C ad E C ita F C ad C G quare C E ad C G
maiorem habebit proportionem quam centum septuagenis primis ad centum

Translated into Latin as *Dimensio circoli* by Jacobus Cremonensis, c. 1450–1460

Illustrated by Piero della Francesca

Available online with other texts by Archimedes

Archimedes: *Κύκλου μέτρησις* (*Measurement of a circle*)



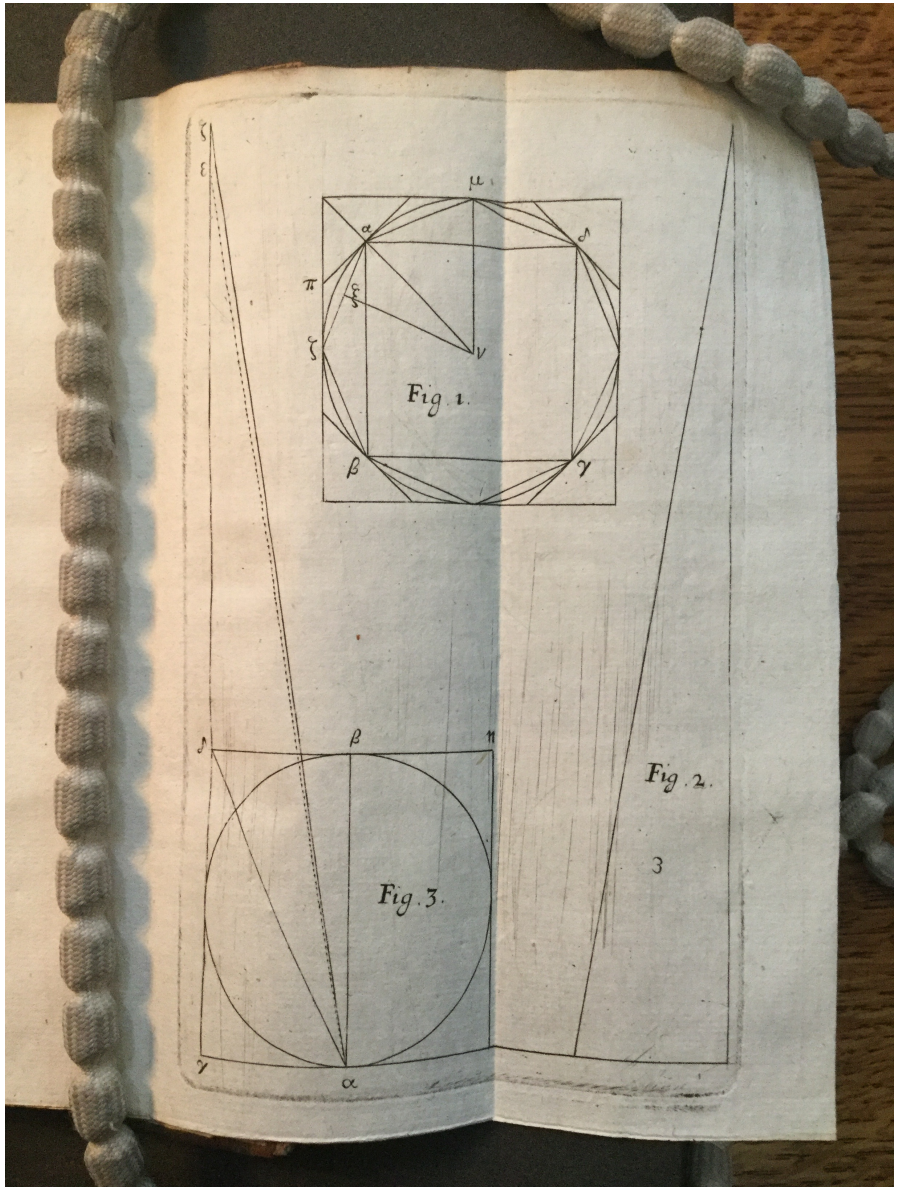
ΡΧΙΜΗΔΟΥΣ
ΤΟΥ
ΣΥΡΑΚΟΥΣΙΟΥ
Ψαμμίτης,
καὶ Κύκλου Μέτρησις
ΕΥΤΟΚΙΟΥ ΑΣΚΑΛΩΝΙΤΟΥ,
εἰς αὐτὴν ὑπόμνημα.

ARCHIMEDIS
SYRACUSANI
Arenarius,
Et Dimensio Circuli.
EUTOCII ASCALONITÆ,
in hanc Commentarius.

Cum Verſione & Notis *Joh. Wallis*, SS.Th.D.
Geometriæ Profeſſoris Saviliani.

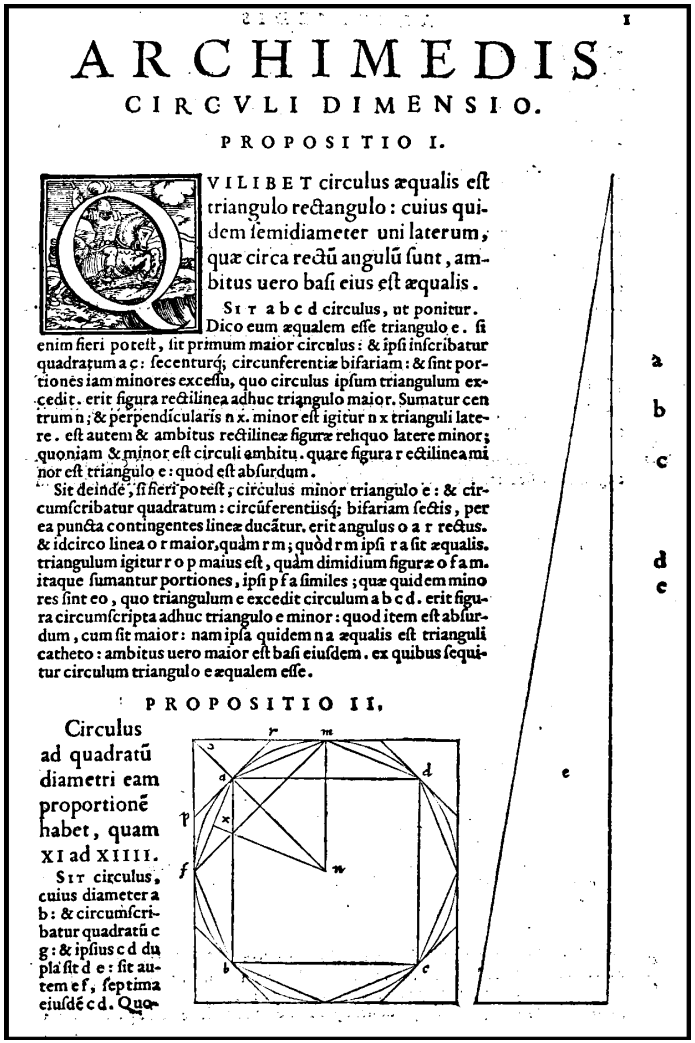
OXONII.
E Theatro SHELDONIANO. 1676.

DICKINSON COLLEGE



Edition by John Wallis, Oxford, 1676

Archimedes: *Κύκλου μέτρησις* (*Measurement of a circle*)



ARCHIMEDIS
CIRCVLI DIMENSIO.

PROPOSITIO I.

QVILIBET circulus æqualis eſt triangulo rectangulo: cuius quidem ſemidiameter uni laterum, quæ circa rectũ angulũ ſunt, ambitus uero baſi eius eſt æqualis.

Sit a b c d circulus, ut ponitur. Dico eum æqualem eſſe triangulo e. ſi enim fieri poteſt, ſit primum maior circulus: & ipſi inſcribatur quadratum a c: ſecenturq; circunferentiæ bifariam: & ſint portiones iam minores exceſſu, quo circulus ipſum triangulum excedit. erit figura rectilinea adhuc triangulo maior. Sumatur centrum n; & perpendicularis n x. minor eſt igitur n x trianguli latere. eſt autem & ambitus rectilineæ figuræ reliquo latere minor; quoniam & minor eſt circuli ambitu. quare figura rectilinea minor eſt triangulo e: quod eſt abſurdum.

Sit deinde, ſi fieri poteſt, circulus minor triangulo e: & circumſcribatur quadratum: circũferentiisq; bifariam ſectis, per ea puncta contingentes lineæ ducãtur. erit angulus o a r rectus. & idcirco linea o r maior, quàm r m; quòd r m ipſi r a ſit æqualis. triangulum igitur r o p maius eſt, quàm dimidium figuræ o f a m. itaque ſumantur portiones, ipſi p f a ſimiles; quæ quidem minores ſint eo, quo triangulum e excedit circulum a b c d. erit figura circumſcripta adhuc triangulo e minor: quod item eſt abſurdum, cum ſit maior: nam ipſa quidem n a æqualis eſt trianguli catheto: ambitus uero maior eſt baſi eiuſdem. ex quibus ſequitur circulum triangulo e æqualem eſſe.

PROPOSITIO II.

Circulus ad quadratũ diametri eam proportionẽ habet, quam XI ad XIIII.

Sit circulus, cuius diameter a b: & circumſcribatur quadratũ c g: & ipſius c d quadrantem bifariam ſecet d e: ſit autem e f, ſeptima eiuſdẽ c d. Quo-

a
b
c
d
e

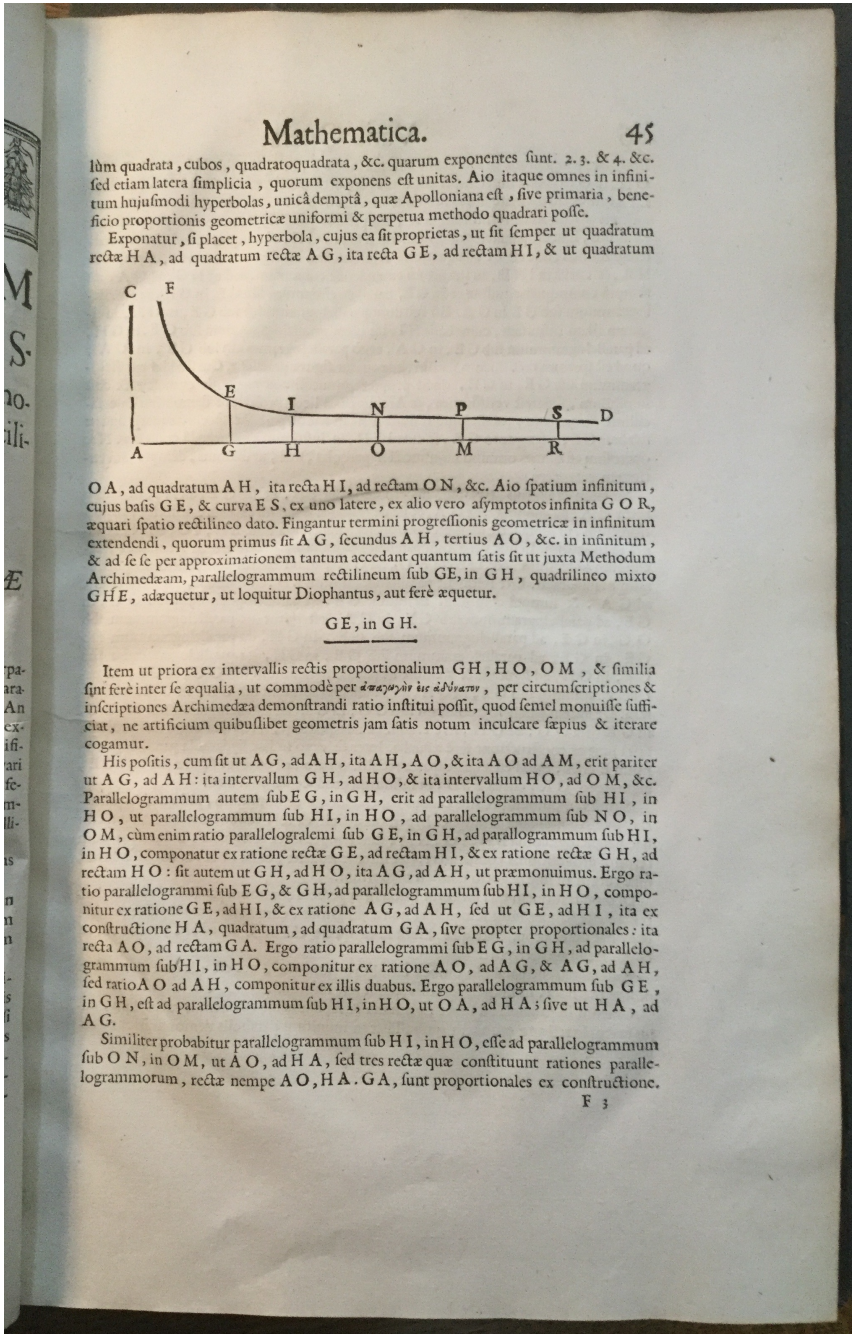
(*Archimedis opera*, edited by Commandino, 1558) — see *Mathematics emerging*, §1.2.3

A circle is equal to a right-angled triangle with height equal to the circumference of the circle and base equal to the radius.

Proof by exhaustion and double contradiction

Later: the ratio of the circumference to the diameter is greater than $3\frac{10}{71}$ and less than $3\frac{1}{7}$.

Fermat's quadrature of a hyperbola (c. 1636)



Mathematica. 45

lùm quadrata, cubos, quadratoquadrata, &c. quarum exponentes ſunt. 2. 3. & 4. &c. ſed etiam latera ſimplicia, quorum exponens eſt unitas. Aio itaque omnes in infinitum hujuſmodi hyperbolas, unicâ demptâ, quæ Apolloniana eſt, ſive primaria, beneficio proportionis geometricæ uniformi & perpetua methodo quadrari poſſe.

Exponatur, ſi placet, hyperbola, cujus ea ſit proprietas, ut ſit ſemper ut quadratum rectæ H A, ad quadratum rectæ A G, ita recta G E, ad rectam H I, & ut quadratum

O A, ad quadratum A H, ita recta H I, ad rectam O N, &c. Aio ſpatium infinitum, cujus baſis G E, & curva E S, ex uno latere, ex alio vero aſymptotos infinita G O R, æquari ſpatio rectilineo dato. Fingantur termini progreſſionis geometricæ in infinitum extendendi, quorum primus ſit A G, ſecundus A H, tertius A O, &c. in infinitum, & ad ſe ſe per approximationem tantum accedant quantum ſatis ſit ut juxta Methodum Archimedæam, parallelogrammum rectilineum ſub GE, in G H, quadrilineo mixto G H E, adæquetur, ut loquitur Diophantus, aut ferè æquetur.

GE, in G H.

Item ut priora ex intervallis rectis proportionalium G H, H O, O M, & ſimilia ſint ferè inter ſe æqualia, ut commodè per ἀπαγωγὴν εἰς ἀδύνατον, per circumſcriptiones & inſcriptiones Archimedæa demonſtrandi ratio inſtitui poſſit, quod ſemel monuiſſe ſufficiat, ne artificium quibuſlibet geometris jam ſatis notum inculcare ſæpius & iterare cogamur.

His poſitis, cum ſit ut A G, ad A H, ita A H, A O, & ita A O ad A M, erit pariter ut A G, ad A H: ita intervallum G H, ad H O, & ita intervallum H O, ad O M, &c. Parallelogrammum autem ſub E G, in G H, erit ad parallelogrammum ſub H I, in H O, ut parallelogrammum ſub H I, in H O, ad parallelogrammum ſub N O, in O M, cùm enim ratio parallelogralemi ſub G E, in G H, ad parallogrammum ſub H I, in H O, componatur ex ratione rectæ G E, ad rectam H I, & ex ratione rectæ G H, ad rectam H O: ſit autem ut G H, ad H O, ita A G, ad A H, ut præmonuimus. Ergo ratio parallelogrammi ſub E G, & G H, ad parallelogrammum ſub H I, in H O, componitur ex ratione G E, ad H I, & ex ratione A G, ad A H, ſed ut G E, ad H I, ita ex conſtructione H A, quadratum, ad quadratum G A, ſive propter proportionales: ita recta A O, ad rectam G A. Ergo ratio parallelogrammi ſub E G, in G H, ad parallelogrammum ſub H I, in H O, componitur ex ratione A O, ad A G, & A G, ad A H, ſed ratio A O ad A H, componitur ex illis duabus. Ergo parallelogrammum ſub G E, in G H, eſt ad parallelogrammum ſub H I, in H O, ut O A, ad H A; ſive ut H A, ad A G.

Similiter probabitur parallelogrammum ſub H I, in H O, eſſe ad parallelogrammum ſub O N, in O M, ut A O, ad H A, ſed tres rectæ quæ conſtituunt rationes parallelogrammorum, rectæ nempe A O, H A. G A, ſunt proportionales ex conſtructione.

F 3

Worked out c. 1636, but only published posthumously in *Varia opera mathematica*, 1679.

In modern terms, this is the curve described by $y = \frac{1}{x^2}$.

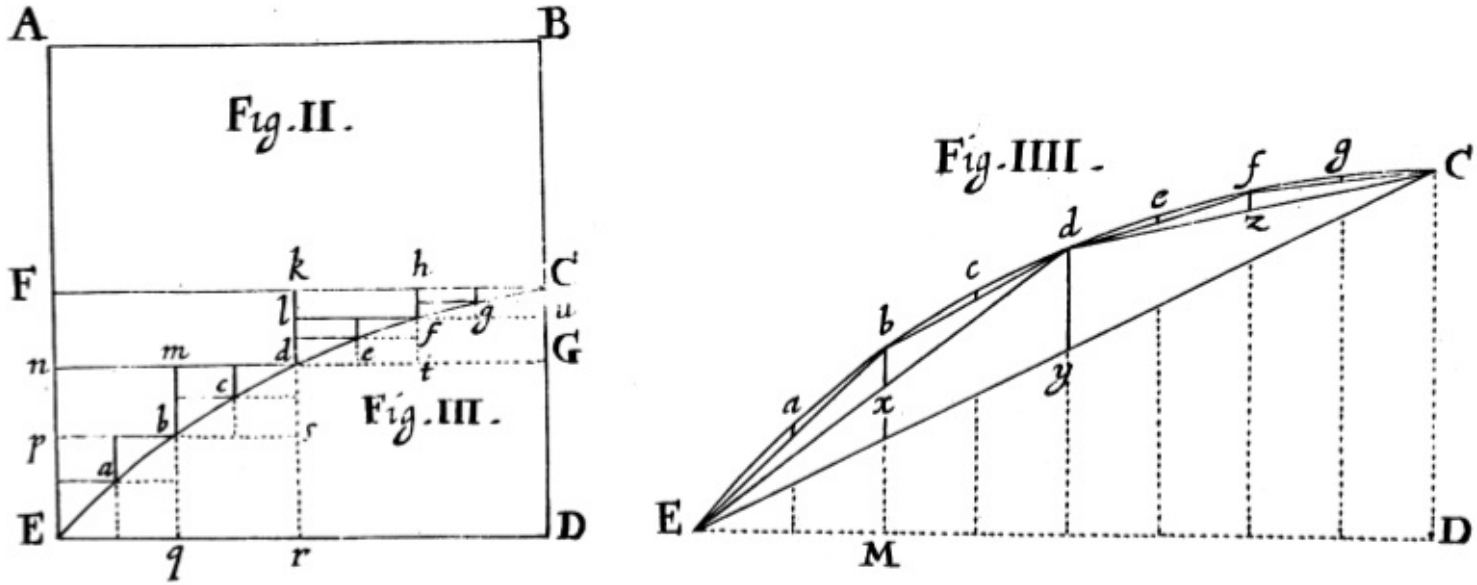
See *Mathematics emerging*, §3.2.1.

The rectangular (or 'Apollonian') hyperbola

In modern notation, $y = \frac{1}{x}$

- Quadrature evaded Fermat
- Partial results obtained by Grégoire de Saint Vincent, c. 1625, published in *Opus geometricum*, 1647
- Empirical observation that if $A(x)$ is the area under the hyperbola from 1 to x, then $A(\alpha\beta) = A(\alpha) + A(\beta)$ (cf. logarithms)
- Problem solved in early 1650s by William Brouncker; published in 1668 in volume 3 of *Philosophical Transactions of the Royal Society*

Brouncker's quadrature of the hyperbola (1668)



To put this into modern terms, take A as the origin, and AB, AE as the x- and y-axes, respectively. Then the diagram represents the area under $\dfrac{1}{1+x}$ from $x = 0$ to $x = 1$.

(See *Mathematics emerging*, §3.2.2.)

Brouncker's article of 1668

PHILOSOPHICAL TRANSACTIONS.

Monday, *April* 13. 1668

The Contents.

The Squaring of the Hyperbola *by an infinite ſeries of* Rational *Numbers, together with its Demonſtration, by the Right Honourable the Lord Viſcount* Brouncker. *An Extract of a Letter ſent from* Danzick, *touching ſome Chymical, Medicinal and Anatomical particulars. Two Letters, written by Dr.* John Wallis *to the* Publiſher; *One, concerning the Variety of the Annual High-Tides in reſpect to ſeveral places: the other, concerning ſome Miſtakes of a Book entituled SPECIMINA MATHEMATICA* Franciſci Dulaurens, *eſpecially touching a certain* Probleme, *affirm'd to have been propoſed by Dr.* Wallis *to the Mathematicians of all* Europe, *for a ſolution. An Account of ſome Obſervations concerning the* true Time *of the Tydes, by Mr.* Hen. Philips. *An Account of three Books: I. W. SENGWERDIUS PH.D.* de Tarantula. *II. REGNERI* de *GRAEF M.D.* Epiſtola de nonnullis circa Partes *Genitales* Inventis Novis. *III. JOHANNIS* van *HORNE M.D.* Obſervationum ſuarum circa Partes *Genitales* in utroque ſexu, *PRODROMUS.*

The Squaring of the Hyperbola, *by an infinite ſeries of Rational Numbers, together with its Demonſtration, by that Eminent Mathematician, the Right Honourable the Lord Viſcount* Brouncker.

WHat the Acute Dr. *John Wallis* had intimated, ſome years ſince, in the Dedication of his Anſwer to M. *Meibomius de proportionibus, vid.* That the World one day would learn from the Noble Lord *Brounker*, the *Quadrature* of the *Hyperbole*; the Ingenious Reader may ſee performed in the ſubjoyned operation, which its Excellent Author was now pleaſed to communicate, as followeth in his own words;

My Method for Squaring the Hyperbola *is this:*

LEt AB be one *Aſymptote* of the Hyperbola EdC; and let AE and BC be parallel to th'other: Let alſo AE be to BC as 2 to 1; and let the Parallelogram ABDE equal 1. See *Fig.* 1. And note, that the Letter x every where ſtands for Multiplication.

Suppoſing the Reader knows, that EA. αζ. KH. βν. dℓ. γκ. ϑλ. ερ. CB. &c. are in an *Harmonic* ſeries, or a *ſeries reciproca primanorum ſeu arithmetice proportionalium* (otherwiſe he is referr'd for ſatisfaction to the 87, 88, 89, 90, 91, 92, 93, 94, 95, prop. *Arithm. Infinitor. Walliſij*:)

$$\text{I ſay } ABCdEA = \frac{1}{1\times2} + \frac{1}{3\times4} + \frac{1}{5\times6} + \frac{1}{7\times8} + \frac{1}{9\times10} \text{ \&c.}$$

$$EdCDE = \frac{1}{2\times3} + \frac{1}{4\times5} + \frac{1}{6\times7} + \frac{1}{8\times9} + \frac{1}{10\times11} \text{ \&c.} \quad \left.\right\} \textit{in infinitum.}$$

$$EdCyE = \frac{1}{2\times3\times4} + \frac{1}{4\times5\times6} + \frac{1}{6\times7\times8} + \frac{1}{8\times9\times10} \text{ \&c.}$$

For (in Fig. 2, & 3) the Parallelog. | *And (in Fig. 4.) the Triangl.*

$$CA = \frac{1}{1\times2}$$

$$dD = \frac{1}{2\times3} \quad dF = \frac{1}{3\times4} \qquad EdC = \frac{1}{2\times3\times4} = \frac{\square dD - \square dF}{2}$$

$$br = \frac{1}{4\times5} \quad bn = \frac{1}{5\times6} \qquad Ebd = \frac{1}{4\times5\times6} = \frac{\square br - \square bn}{2}$$

$$fG = \frac{1}{6\times7} \quad fk = \frac{1}{7\times8} \qquad dfC = \frac{1}{6\times7\times8} = \frac{\square fG - \square fk}{2}$$

$$aq = \frac{1}{8\times9} \quad ap = \frac{1}{9\times10} \qquad Eab = \frac{1}{8\times9\times10} = \frac{\square aq - \square ap}{2}$$

$$cs = \frac{1}{10\times11} \quad cm = \frac{1}{11\times12} \qquad bcd = \frac{1}{10\times11\times12} = \frac{\square cs - \square cm}{2}$$

$$et = \frac{1}{12\times13} \quad el = \frac{1}{13\times14} \qquad def = \frac{1}{12\times13\times14} = \frac{\square et - \square el}{2}$$

$$gu = \frac{1}{14\times15} \quad gh = \frac{1}{15\times16} \qquad fgC = \frac{1}{14\times15\times16} = \frac{\square gu - \square gh}{2}$$

&c. &c. &c.

Note.

CA = dD + dF

dD = br + bn

dF = fG + fk

br = aq + ap

bn = cs + cm

fG = et + el

fk = gu + gh

&c.

New methods: indivisibles and infinitesimals

Indivisibles: geometric objects making up a higher-dimensional object (e.g., points $\rightarrow$ line, lines $\rightarrow$ plane)

Infinitesimal: arbitrarily small but nonzero quantity

But distinction often blurred

During the 17th century, both concepts saw much use — despite the fact that they appeared to contradict Euclidean principles

Indivisibles

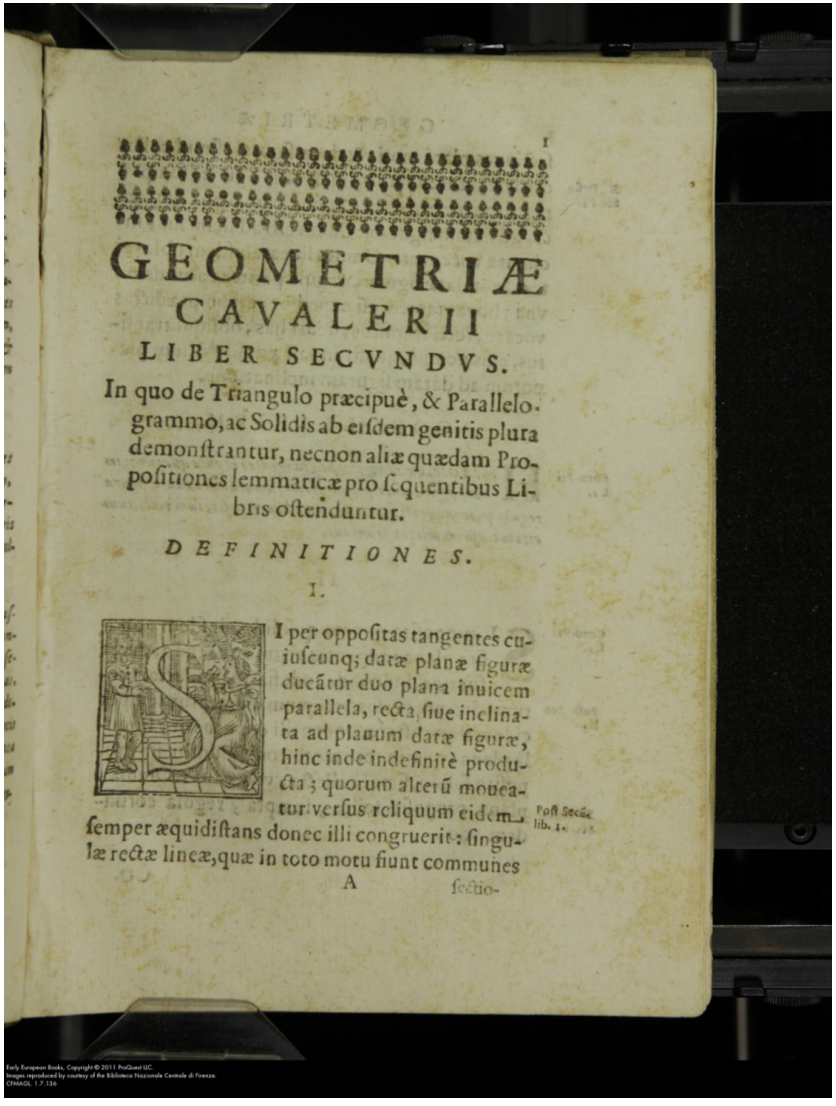
Early treatments by de Saint Vincent in c. 1623 (but not published until 1647) and Roberval in c. 1628–34 (but not published until 1693).

First published treatment by Bonaventura Cavalieri (1598–1647) in *Geometria indivisibilibus continuorum nova quadam ratione promota* [*Geometry advanced in a new way by the indivisibles of the continua*] (1635).

Used by Evangelista Torricelli (1608–1647) in 1644 to calculate the volume of an infinite hyperboloid of revolution.

Developed by John Wallis (1616–1703) and others.

Cavalieri's *Geometria*



I

GEOMETRIÆ CAVALERII

LIBER SECVNDVS.

In quo de Triangulo præcipuè, & Parallelogrammo, ac Solidis ab eisdem genitis plura demonstrantur, necnon aliæ quædam Propositiones lemmaticæ pro sequentibus Libris ostenduntur.

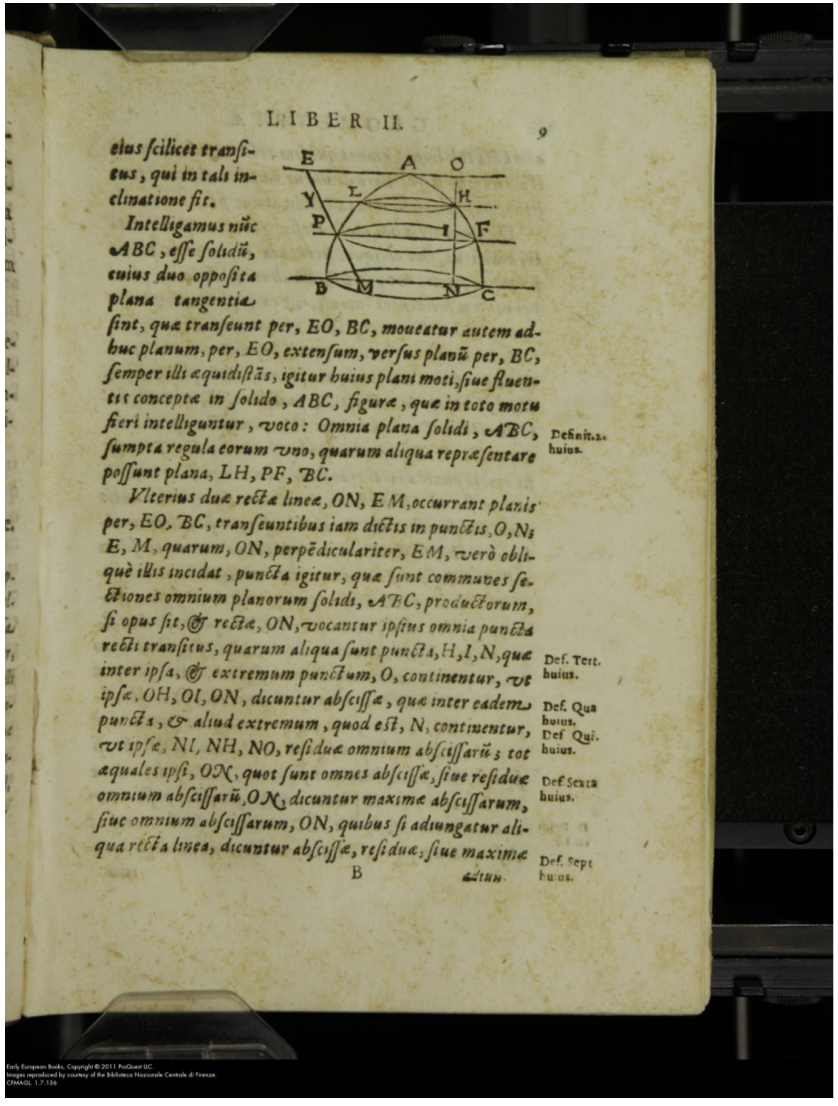
DEFINITIONES.

I.

SI per oppositas tangentes cuiuscunq; datæ planæ figuræ ducãtur duo plana inuicem parallela, recta, siue inclinata ad planum datæ figuræ, hinc inde indefinitè producta; quorum alterũ moueatur versus reliquum eidem semper æquidistans donec illi congruerit: singulæ rectæ lineæ, quæ in toto motu fiunt communes

Post Sectã lib. 1.

A sectio-



eius scilicet transitus, qui in tali inclinatione fit.

Intelligamus nũc ABC, esse solidũ, cuius duo opposita plana tangentia sint, quæ transeunt per, EO, BC, moueatur autem adhuc planum, per, EO, extensum, versus planũ per, BC, semper illi æquidistãs, igitur huius plani moti, siue fluentis conceptæ in solido, ABC, figuræ, quæ in toto motu fieri intelliguntur, voco: Omnia plana solidi, ABC, sumpta regula eorum vno, quarum aliqua repræsentare possunt plana, LH, PF, BC. Definit. 2. huius.

Vlterius duæ rectæ lineæ, ON, EM, occurrant planis per, EO, BC, transeuntibus iam dictis in punctis, O, N; E, M, quarum, ON, perpẽdiculariter, EM, verò obliquè illis incidat, puncta igitur, quæ sunt communes sectiones omnium planorum solidi, ABC, productorum, si opus sit, & rectæ, ON, vocantur ipsius omnia puncta recti transitus, quarum aliqua sunt puncta, H, I, N, quæ inter ipsa, & extremum punctum, O, continentur, vt ipsæ, OH, OI, ON, dicuntur abscissæ, quæ inter eadem puncta, & aliud extremum, quod est, N, continentur, vt ipsæ, NI, NH, NO, residuæ omnium abscissarũ; tot æquales ipsi, ON, quot sunt omnes abscissæ, siue residuæ omnium abscissarũ, ON, dicuntur maximæ abscissarum, siue omnium abscissarum, ON, quibus si adiungatur aliqua recta linea, dicuntur abscissæ, residuæ, siue maximæ Def. Tert. huius. Def. Qua huius. Def. Qui. huius. Def. Sexta huius. Def. Sept huius.

B *alius*

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Torricelli's hyperbolic solid (*Opera geometrica*, 1644)

Problema Secundum. 115

Lemma V.

CVIuſcunque cylindri ghil intra ſolidum acutum deſcripti (vt in precedenti figura) ſuperficies ſine baſibus æqualis eſt circulo cuius ſemidiameter ſit linea df. *nempe ſemiaxis, ſiue ſemilatus verſum ipſius hyperbolæ. Hoc enim in ipſo progreſſu precedentis lemmatis demonſtratum eſt.*

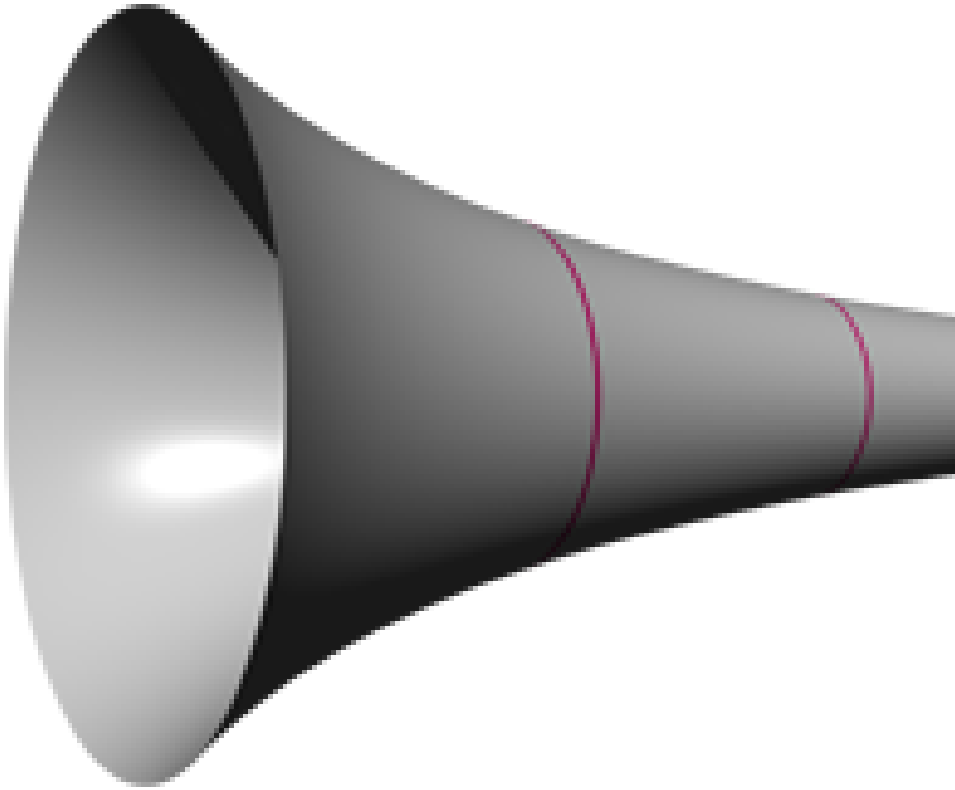
Theorema.

SOlidum acutum hyperbolicum infinitè longum, ſectum plano ad axem erecto, vnà cum cylindro ſuæ baſis, æquale eſt cylindro cuidam recto, cuius baſis diameter ſit latus verſum, ſiue axis hyperbolæ, altitudo verò ſit æqualis ſemidiametro baſis ipſius acuti ſolidi.

Eſto hyperbola cuius aſymptoti *ab*, *ac* angulum rectum contineant; ſumptoq; in hyperbola quolibet puncto *d*, ducatur *dc* æquidiſtans ipſi *ab*, & *dp* æquidiſtans *ac*. Tũ conuertatur vniuerſa figura circa axẽ *ab*. ità vt fiat ſolidum acutum byperbolicum *ebd*, vna cum cylindro ſuæ baſis *fedc*. Poducatur *ba* in *h*. ita vt *ah* æqualis ſit integro axi, ſiue lateri verſo hyperbolæ. Et circa diametrum *ah* intelligatur circulus erectus ad aſymptoton *ac*: & ſuper baſi *ah* concipiatur cylindrus rectus *acgh*, cuius altitudo ſit *ac*, nempe ſemidiameter baſis acuti ſolidi. Dico ſolidum vniuerſum *febdc*, quanquam ſine fine longum, æquale tamen eſſe cylindro *acgh*.

Accipiatur in recta *ac* quodlibet punctum *i*, & per *i* intelligatur ducta ſuperficies cylindrica *onli* in ſolido acuto com-

P 2



(See *Mathematics emerging*, §3.3.1.)

John Wallis (1616–1703)

Studied at Emmanuel College, Cambridge (BA 1637, MA 1640)

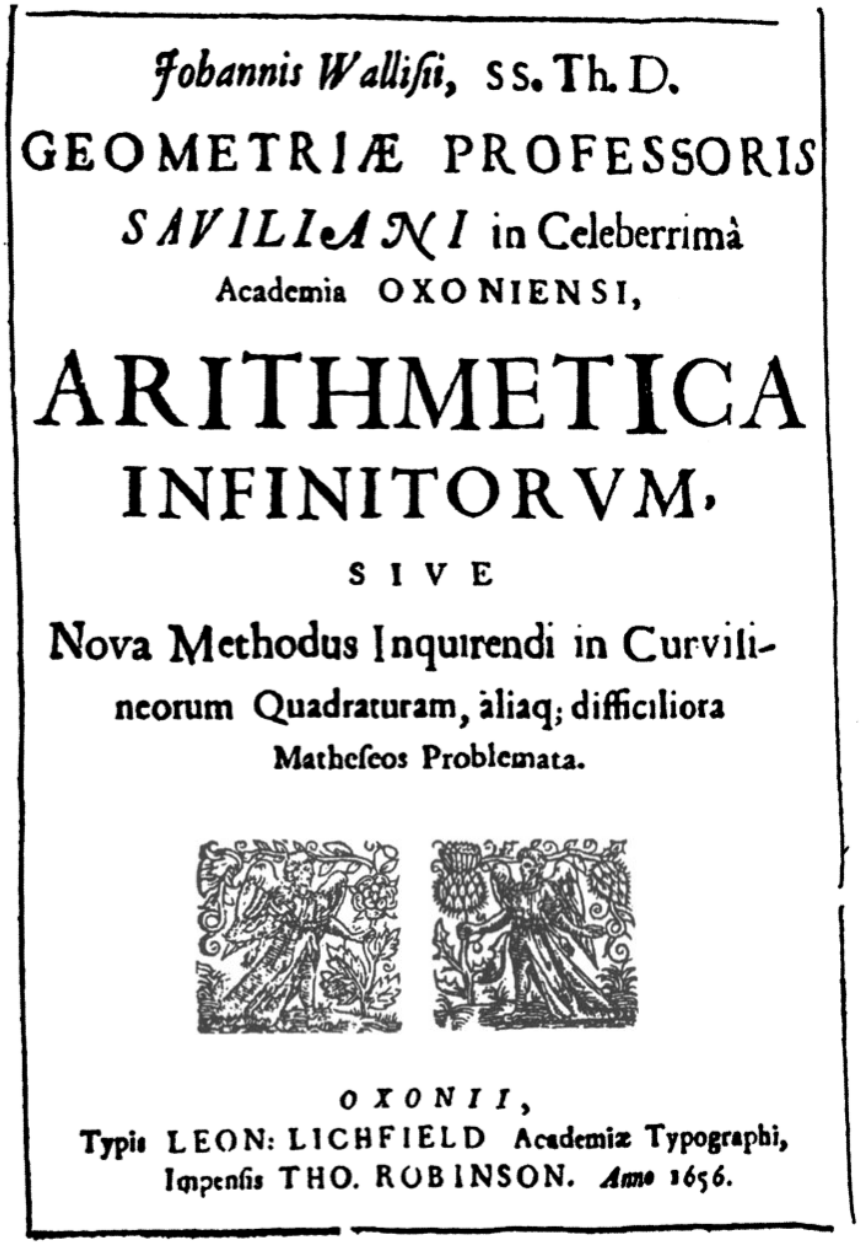
1643–1649: scribe for Westminster Assembly

1644–1645: Fellow of Queens' College, Cambridge

1643–1689: cryptographer to Parliament, then to the Crown

1649–1703: Savilian Professor of Geometry in Oxford

Arithmetica infinitorum



Johannis Wallisii, SS. Th. D.
GEOMETRIÆ PROFESSORIS
SAVILIANI in Celeberrimâ
Academia OXONIENSI,

ARITHMETICA
INFINITORVM,

SIVE

Nova Methodus Inquirendi in Curvilineorum Quadraturam, aliaq; difficiliora Matheseos Problemata.

OXONII,
Typis LEON: LICHFIELD Academiæ Typographi,
Impensis THO. ROBINSON. *Anno* 1656.

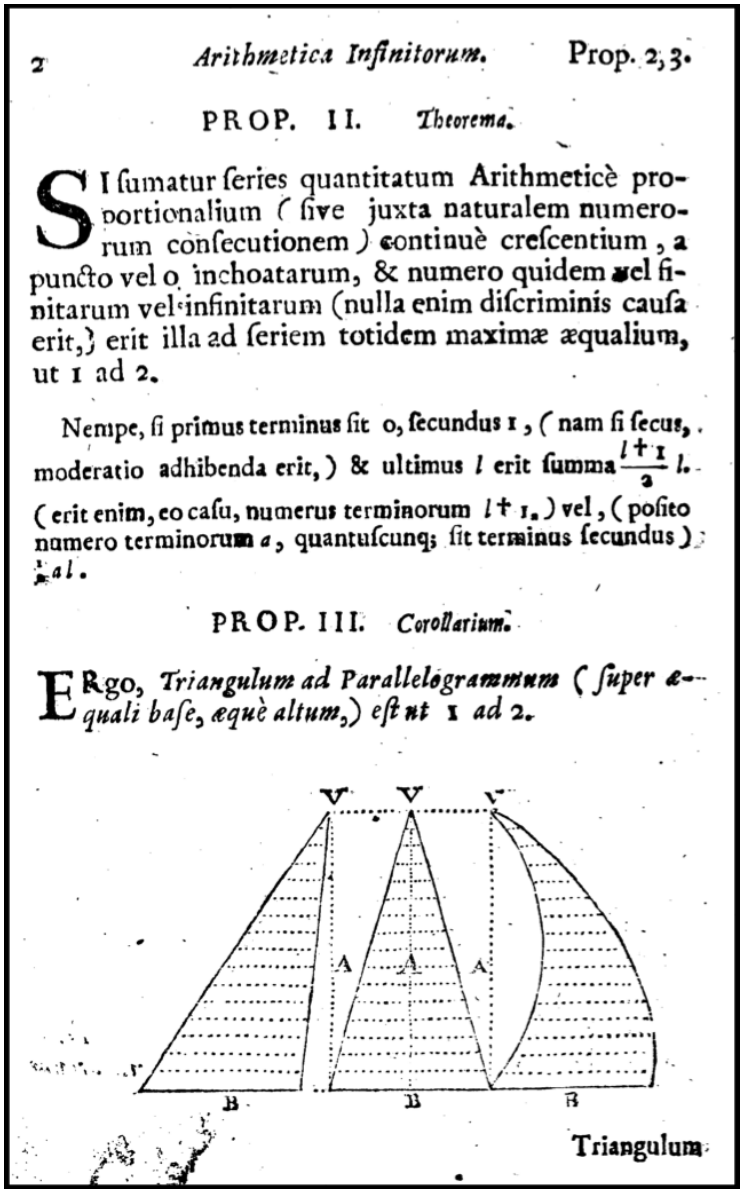
John Wallis,
Arithmetica infinitorum
(*The arithmetic of infinitesimals*)
Oxford, 1656

Translation by
Jacqueline A. Stedall
Springer, 2004

Arithmetica infinitorum

- Arithmetical methods rather than geometrical, but repeatedly appealed to geometry for justification
- Investigation of sums of sequences of powers (or ratios of these to a known fixed quantity) — usually decreasing
- Fixed an endpoint, dividing interval into infinite number of arbitrarily small subintervals — these are the 'infinitesimals' of Wallis' title

Wallis and indivisibles



2 *Arithmetica Infinitorum.* Prop. 2, 3.

PROP. II. *Theorema.*

SI ſumatur ſeries quantitatum Arithmeticè proportionalium (ſive juxta naturalem numerorum conſecutionem) continuè creſcentium, a puncto vel 0 inchoatarum, & numero quidem vel finitarum vel infinitarum (nulla enim diſcriminis cauſa erit,) erit illa ad ſeriem totidem maximæ æqualium, ut 1 ad 2.

Nempe, ſi primus terminus ſit 0, ſecundus 1, (nam ſi ſecus, moderatio adhibenda erit,) & ultimus l erit ſumma $\frac{l+1}{2}\,l$. (erit enim, eo caſu, numerus terminorum $l+1$.) vel, (poſito numero terminorum a, quantuſcunq; ſit terminus ſecundus) $\frac{1}{2}\,al$.

PROP. III. *Corollarium.*

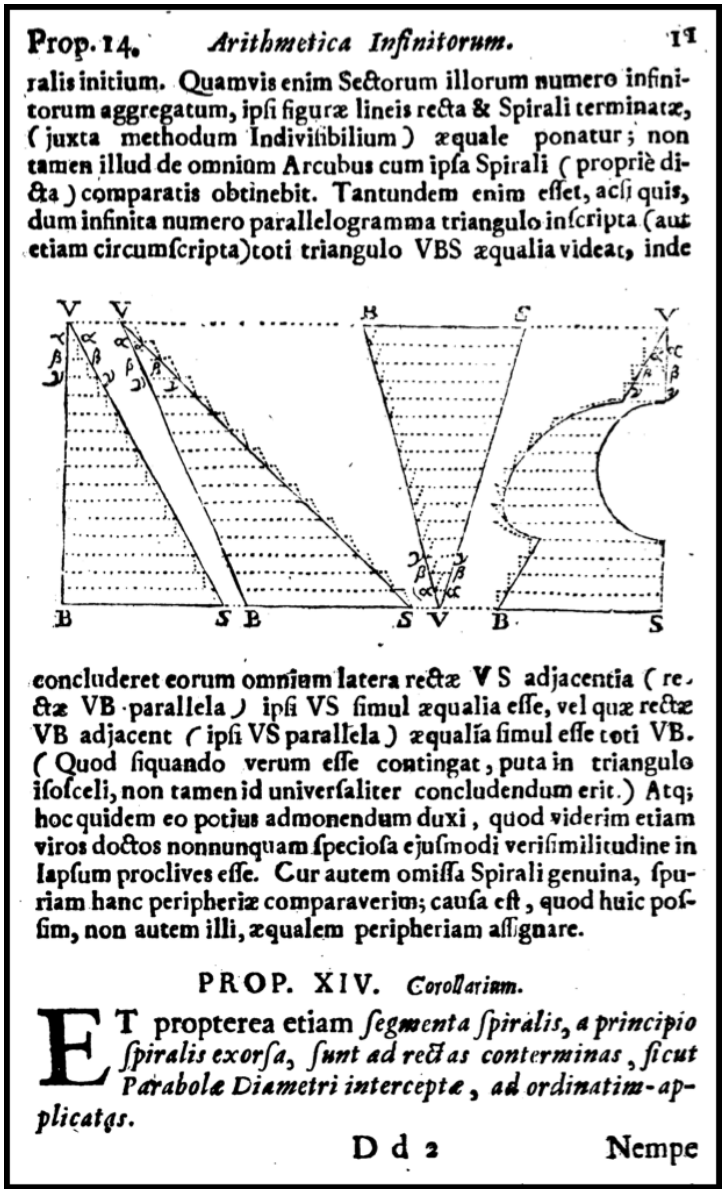
ERgo, *Triangulum ad Parallelogrammum (ſuper æquali baſe, æquè altum,) eſt ut 1 ad 2.*

Triangulum

For the triangle . . . consists of an infinite number of parallel lines in arithmetic proportion . . .

(See *Mathematics emerging*, §2.4.2.)

Wallis and indivisibles?



Prop. 14. *Arithmetica Infinitorum.* 191

ralis initium. Quamvis enim Sectorum illorum numero infinitorum aggregatum, ipſi figuræ lineis recta & Spirali terminatæ, (juxta methodum Indiviſibilium) æquale ponatur; non tamen illud de omnium Arcubus cum ipſa Spirali (propriè dicta) comparatis obtinebit. Tantundem enim eſſet, acſi quis, dum infinita numero parallelogramma triangulo inſcripta (aut etiam circumſcripta) toti triangulo VBS æqualia videat, inde

concluderet eorum omnium latera rectæ VS adjacentia (rectæ VB parallela) ipſi VS ſimul æqualia eſſe, vel quæ rectæ VB adjacent (ipſi VS parallela) æqualia ſimul eſſe toti VB. (Quod ſiquando verum eſſe contingat, puta in triangulo iſoſceli, non tamen id univerſaliter concludendum erit.) Atq; hoc quidem eo potius admonendum duxi, quod viderim etiam viros doctos nonnunquam ſpecioſa ejuſmodi veriſimilitudine in lapſum proclives eſſe. Cur autem omiſſa Spirali genuina, ſpuriam hanc peripheriæ comparaverim; cauſa eſt, quod huic poſſim, non autem illi, æqualem peripheriam aſſignare.

PROP. XIV. *Corollarium.*

ET propterea etiam *ſegmenta ſpiralis, a principio ſpiralis exorſa, ſunt ad rectas conterminas, ſicut Parabolæ Diametri intercepte, ad ordinatim-applicatas.*

For it amounts to the same thing as if, when an infinite number of parallelograms are inscribed in (or circumscribed about) a triangle, it seems that they equal to complete triangle . . .

(See *Mathematics emerging*, §2.4.2.)

Sums of powers

Wallis' method depended upon the summation rule

$$\sum_{a=0}^{A} a^n \approx \frac{A^{n+1}}{n+1}$$

A version of this was developed by Ibn al-Haytham (Alhazen) in 11th-century Egypt, and it was certainly known to Fermat, Roberval, and Cavalieri in the 1630s for positive integers n, but in the 1650s Wallis extended it to negative and fractional n.

See: Victor J. Katz, Ideas of calculus in Islam and India, *Mathematics Magazine* 68(3) (1995), 163–174

Simple 'integrals'

Using the summation rule we can 'integrate'

$$x^2, \quad x^3, \quad ..., \quad x^{1/3}, \quad ..., \quad x^{-4}, \quad ...$$

and

$$(1+x)^3 \quad \text{or} \quad (1+x^2)^5 \quad \text{or} \quad ...$$

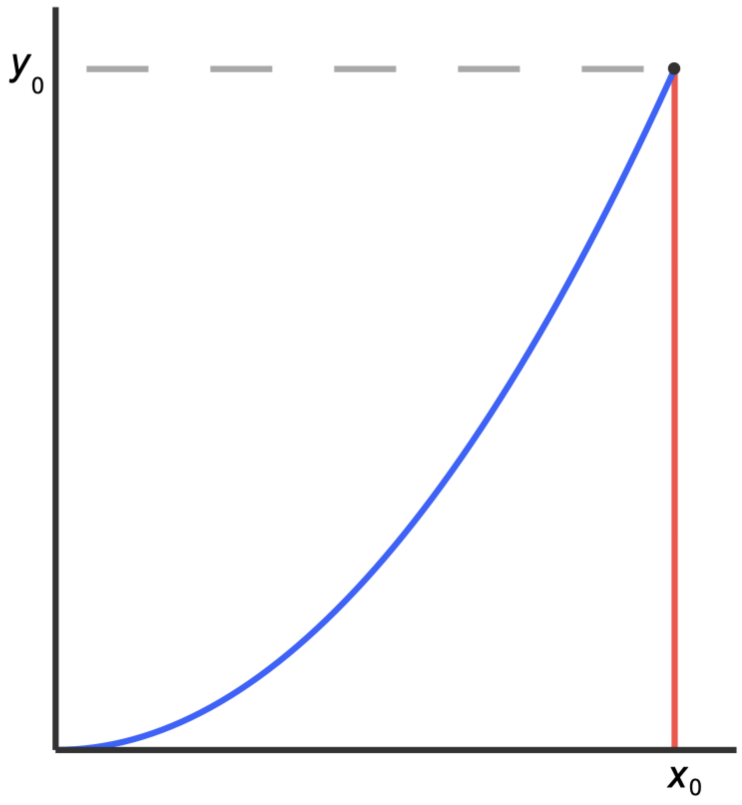
but what about

$$(1-x^2)^{1/2} \quad \text{[for a circle]}$$

or

$$(1+x)^{-1} \quad \text{[for a hyperbola]}\,?$$

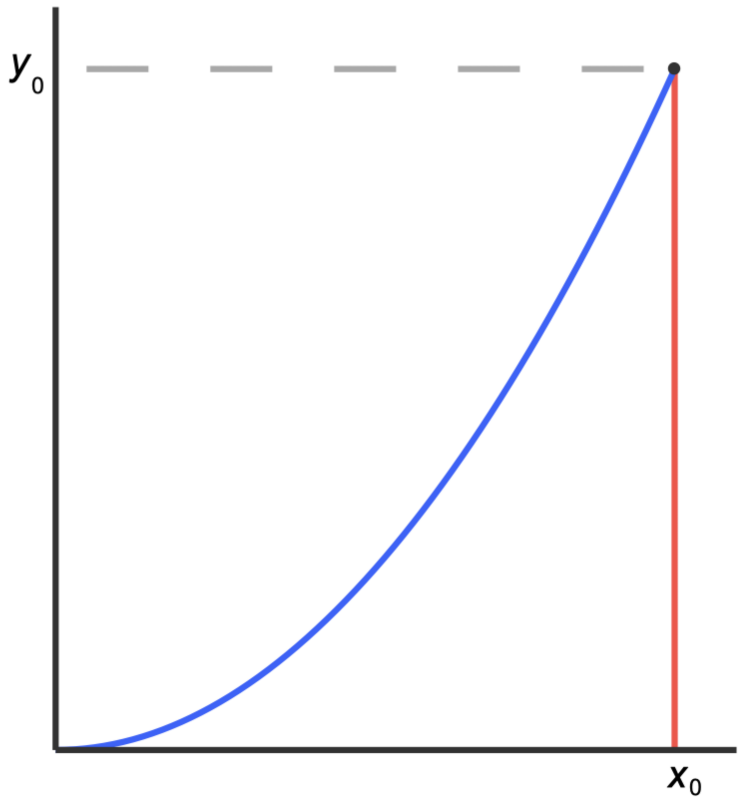
Wallis and the quadrature of a parabola



Wallis sought the area under the parabola $y = x^2$ between $x = 0$ and $x = x_0$

He used the language of ratio, hence sought to calculate the ratio of the area A under the curve to that of the corresponding rectangle $(x_0 y_0)$, which we may think of as the fraction $\frac{A}{x_0 y_0}$

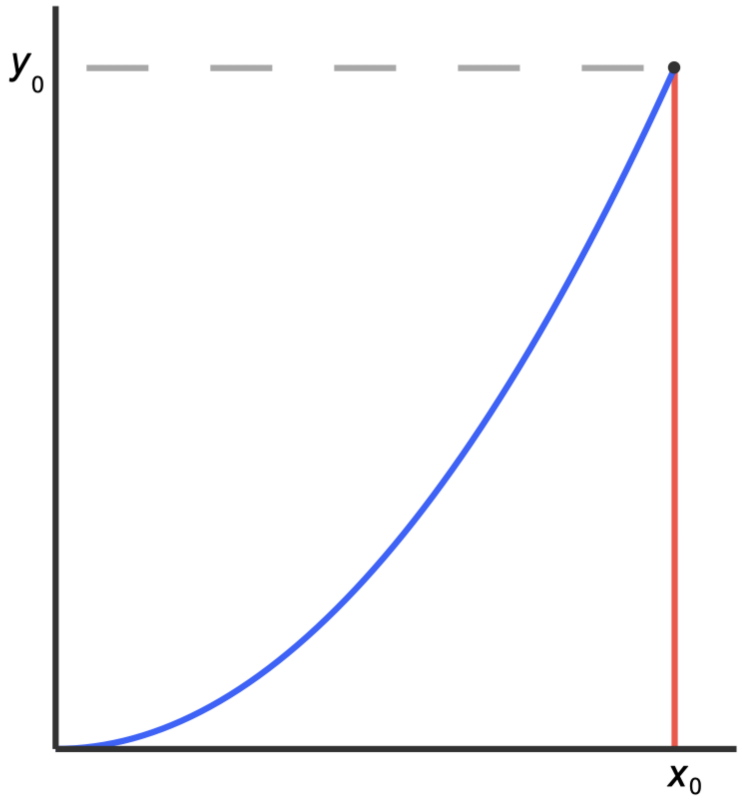
Wallis and the quadrature of a parabola



Wallis considered an area to be the sum of the lengths of the lines contained within it (makes sense?), so

- A is the sum of the values of x^2 as x ranges from 0 to x_0
- $x_0 y_0$ is the sum of as many copies of x_0^2 (?)

Wallis and the quadrature of a parabola



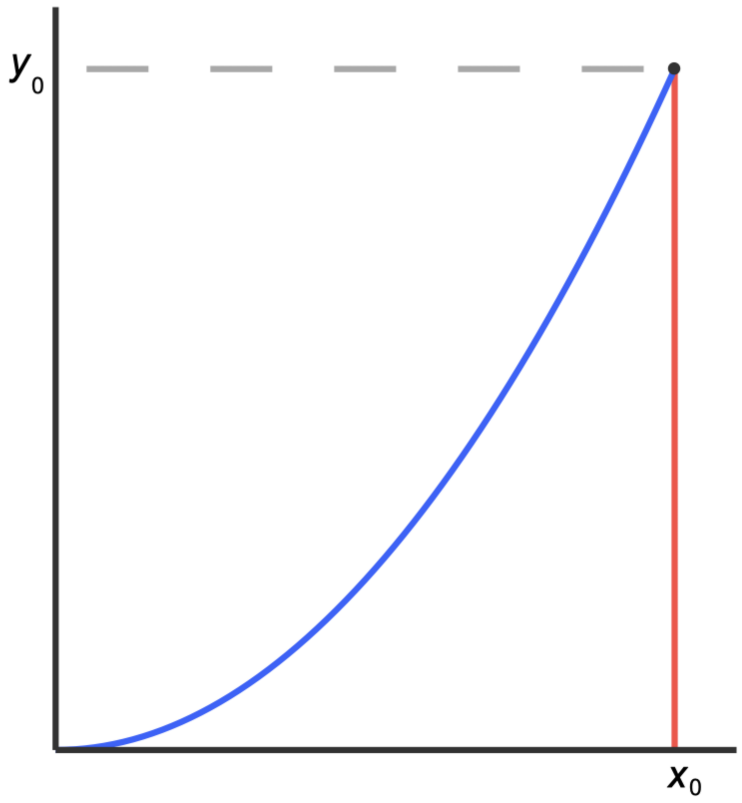
Break $(0, x_0)$ into n subintervals, suppose that x only takes the values at the endpoints of these, and consider the ratio

$$R = \frac{0^2 + 1^2 + 2^2 + \cdots + n^2}{n^2 + n^2 + n^2 + \cdots + n^2}$$

As we make n larger, this ratio will become a closer approximation to $\frac{A}{x_0 y_0}$

[Note that we are deliberately avoiding the terminology of limits, and that some x_0^2s have been cancelled, thanks to the use of ratios]

Wallis and the quadrature of a parabola

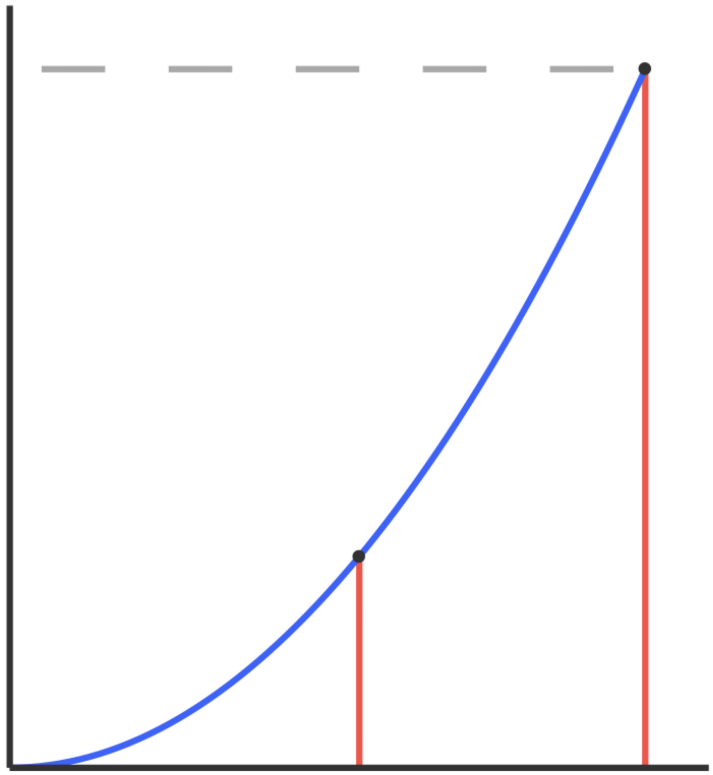


Wallis investigated the cases of small n

For $n = 1$ (one red line),

$$R = \frac{0^2 + 1^2}{1^2 + 1^2} = \frac{1}{2} = \frac{1}{3} + \frac{1}{6}$$

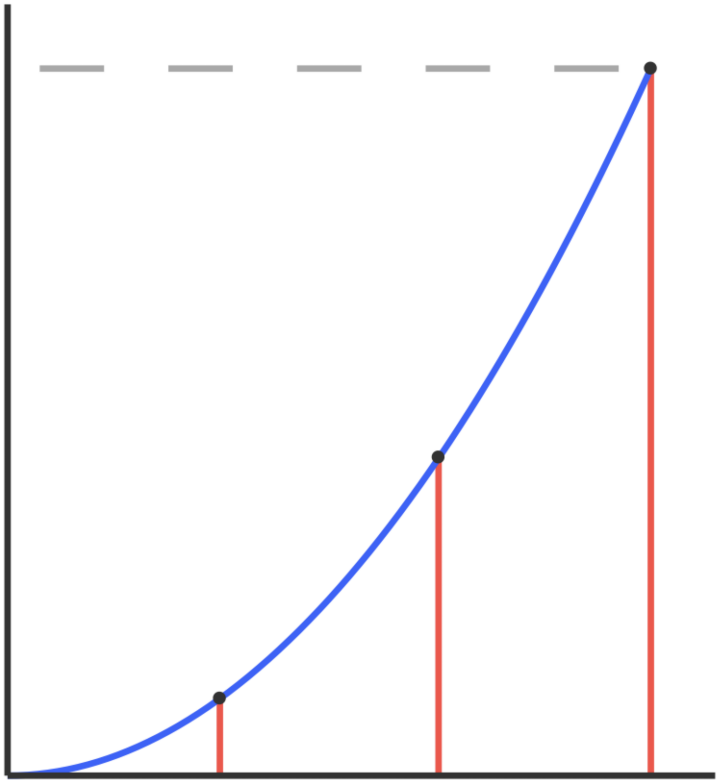
Wallis and the quadrature of a parabola



For $n = 2$ (two red lines),

$$R = \frac{0^2 + 1^2 + 2^2}{2^2 + 2^2 + 2^2} = \frac{5}{12} = \frac{1}{3} + \frac{1}{12}$$

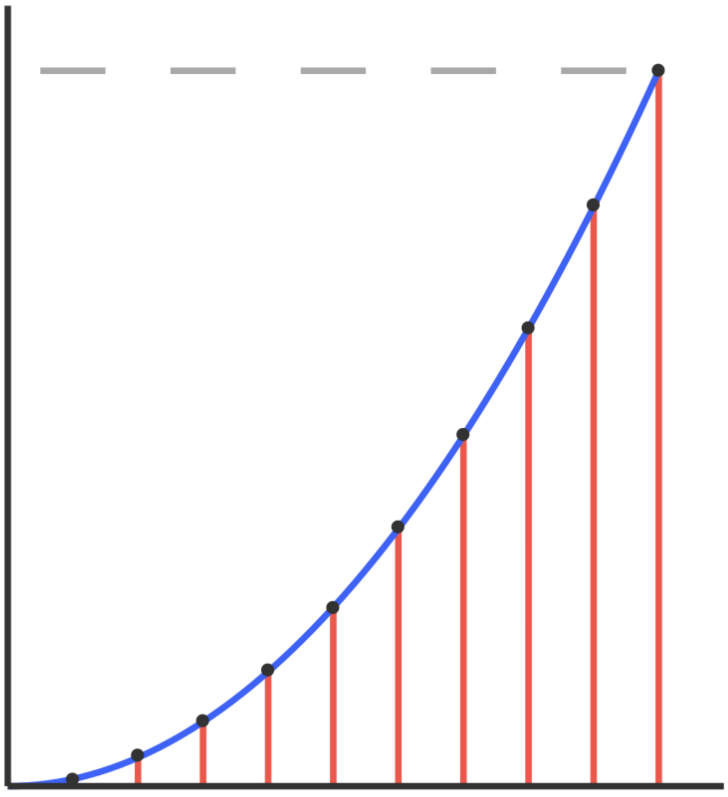
Wallis and the quadrature of a parabola



For $n = 3$ (three red lines),

$$R = \frac{0^2 + 1^2 + 2^2 + 3^2}{3^2 + 3^2 + 3^2 + 3^2}$$
$$= \frac{14}{36} = \frac{1}{3} + \frac{1}{18}$$

Wallis and the quadrature of a parabola



So as n increases $\frac{A}{x_0 y_0}$ approaches $\frac{1}{3}$, hence $A = \frac{1}{3}x_0^3$, as we'd expect

Wallis called this method of spotting and extending a pattern 'induction' — it was criticised at the time (for example, by Pascal)

Enter Newton...

In his own words:

> *In the winter between the years 1664 and 1665 upon reading Dr Wallis's Arithmetica infinitorum and trying to interpole his progressions for squaring the circle, I found out first an infinite series for squaring the circle and then another infinite series for squaring the Hyperbola ...*

Newton extended Wallis' method of interpolation...

Newton's integration of $(1+x)^{-1}$

	$(1+x)^{-1}$	$(1+x)^{0}$	$(1+x)^{1}$	$(1+x)^{2}$	$(1+x)^{3}$	$(1+x)^{4}$	$\cdots$
x	?	1	1	1	1	1	$\cdots$
$\frac{x^2}{2}$	?	0	1	2	3	4	$\cdots$
$\frac{x^3}{3}$	?	0	0	1	3	6	$\cdots$
$\frac{x^4}{4}$	?	0	0	0	1	4	$\cdots$
$\frac{x^5}{5}$	?	0	0	0	0	1	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

The entry in the row labelled $\frac{x^m}{m}$ and the column labelled $(1+x)^n$ is the coefficient of $\frac{x^m}{m}$ in $\int (1+x)^n dx$. (NB. Newton did not use the notation $\int (1+x)^n dx$.)

Newton's integration of $(1+x)^{-1}$

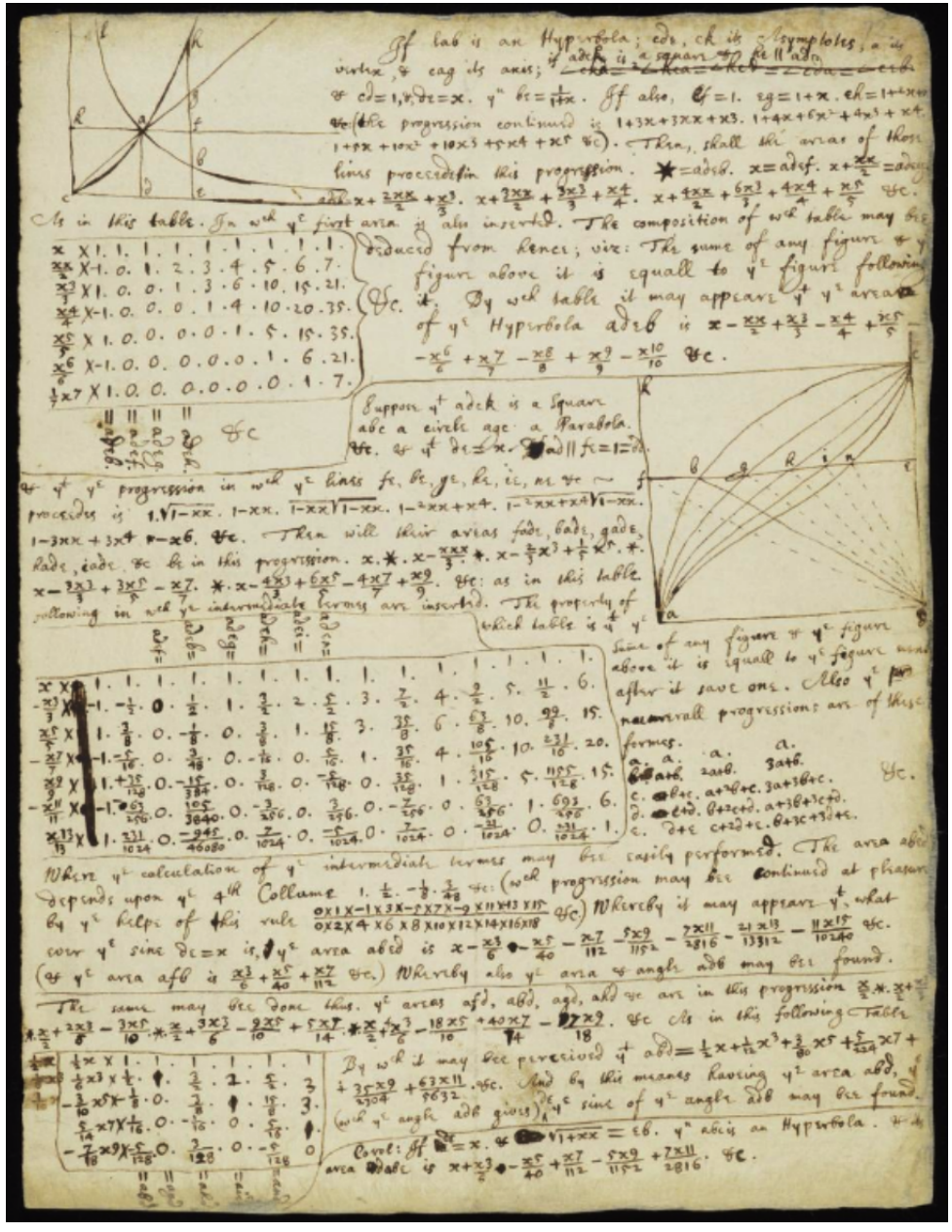
	$(1+x)^{-1}$	$(1+x)^{0}$	$(1+x)^{1}$	$(1+x)^{2}$	$(1+x)^{3}$	$(1+x)^{4}$	$\cdots$
x	1	1	1	1	1	1	$\cdots$
$\frac{x^2}{2}$	-1	0	1	2	3	4	$\cdots$
$\frac{x^3}{3}$	1	0	0	1	3	6	$\cdots$
$\frac{x^4}{4}$	-1	0	0	0	1	4	$\cdots$
$\frac{x^5}{5}$	1	0	0	0	0	1	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

The entry in the row labelled $\frac{x^m}{m}$ and the column labelled $(1+x)^n$ is the coefficient of $\frac{x^m}{m}$ in $\int (1+x)^n dx$. (NB. Newton did not use the notation $\int (1+x)^n dx$.)

The general binomial theorem

CUL Add. MS 3958.3, f. 72

See *Mathematics emerging*, §8.1.1



Newton's calculus: 1664–5

- rules for quadrature (influenced by Wallis's ideas of interpolation)
- rules for tangents (influenced by Descartes' double root method)
- recognition that these are inverse processes

Newton's vocabulary and notation

Newton's calculus 1664–5:

- fluents x, y, ... (quantities that vary with time t)
- fluxions $\dot{x}$, $\dot{y}$, ... (rate of change of those quantities)
- moments o (infinitesimal time in which x increases by $\dot{x}o$)

Newton's calculus in action (*The method of fluxions*, 1736)

12. Ex. 5. As if the Equation $zz + axz - y^4 = 0$ were propos'd to expreſs the Relation between x and y, as alſo $\sqrt{ax - xx} = BD$, for determining a Curve, which therefore will be a Circle. The Equation $zz + axz - y^4 = 0$, as before, will give $2zz\dot{z} + az\dot{x} + ax\dot{z} - 4y^3\dot{y} = 0$, for the Relation of the Celerities $\dot{x}$, $\dot{y}$, and $\dot{z}$. And therefore ſince it is $z = x \times BD$ or $= x\sqrt{ax - xx}$, ſubſtitute this Value inſtead of it, and there will ariſe the Equation $2xz\dot{z} + axx\sqrt{ax - xx} + ax\dot{z} - 4y^3\dot{y} = 0$, which determines the Relation of the Celerities $\dot{x}$ and $\dot{y}$.

DEMONSTRATION *of the Solution.*

13. The Moments of flowing Quantities, (that is, their indefinitely ſmall Parts, by the acceſſion of which, in indefinitely ſmall portions of Time, they are continually increaſed,) are as the Velocities of their Flowing or Increaſing.

14. Wherefore if the Moment of any one, as x, be repreſented by the Product of its Celerity $\dot{x}$ into an indefinitely ſmall Quantity o (that is, by $\dot{x}o$,) the Moments of the others v, y, z, will be repreſented by $\dot{v}o$, $\dot{y}o$, $\dot{z}o$; becauſe $\dot{v}o$, $\dot{x}o$, $\dot{y}o$, and $\dot{z}o$, are to each other as $\dot{v}$, $\dot{x}$, $\dot{y}$, and $\dot{z}$.

See the Analiſe des infiniment petits of the M. de L'Hospital.

15. Now ſince the Moments, as $\dot{x}o$ and $\dot{y}o$, are the indefinitely little acceſſions of the flowing Quantities x and y, by which thoſe Quantities are increaſed through the ſeveral indefinitely little intervals of Time; it follows, that thoſe Quantities x and y, after any indefinitely ſmall interval of Time, become $x + \dot{x}o$ and $y + \dot{y}o$. And therefore the Equation, which at all times indifferently expreſſes the Relation of the flowing Quantities, will as well expreſs the Relation between $x + \dot{x}o$ and $y + \dot{y}o$, as between x and y: So that $x + \dot{x}o$ and $y + \dot{y}o$ may be ſubſtituted in the ſame Equation for thoſe Quantities, inſtead of x and y.

16. Therefore let any Equation $x^3 - ax^2 + axy - y^3 = 0$ be given, and ſubſtitute $x + \dot{x}o$ for x, and $y + \dot{y}o$ for y, and there will ariſe

$$\left.\begin{array}{llll} x^3 & +3\dot{x}ox^2 & +3\dot{x}^2o^2x & +\dot{x}^3o^3 \\ -ax^2 & -2a\dot{x}ox & -a\dot{x}^2o^2 & \\ +axy & +a\dot{x}oy & +ay\dot{x}o & +a\dot{x}y\dot{x}o^2 \\ -y^3 & -3\dot{y}oy^2 & -3\dot{y}^2o^2y & -\dot{y}^3o^3 \end{array}\right\} = 0.$$

17.

17. Now by Suppoſition $x^3 - ax^2 + axy - y^3 = 0$, which therefore being expunged, and the remaining Terms being divided by o, there will remain $3\dot{x}x^2 + 3\dot{x}^2ox + \dot{x}^3o^2 - 2a\dot{x}x - a\dot{x}^2o + a\dot{x}y + ay\dot{x} + a\dot{x}yo - 3\dot{y}y^2 - 3\dot{y}^2oy - \dot{y}^3o^2 = 0$. But whereas o is ſuppoſed to be infinitely little, that it may repreſent the Moments of Quantities; the Terms that are multiply'd by it will be nothing in reſpect of the reſt. Therefore I reject them, and there remains $3\dot{x}x^2 - 2a\dot{x}x + a\dot{x}y + ay\dot{x} - 3\dot{y}y^2 = 0$, as above in Examp. 1.

18. Here we may obſerve, that the Terms that are not multiply'd by o will always vaniſh, as alſo thoſe Terms that are multiply'd by o of more than one Dimenſion. And that the reſt of the Terms being divided by o, will always acquire the form that they ought to have by the foregoing Rule: Which was the thing to be proved.

19. And this being now ſhewn, the other things included in the Rule will eaſily follow. As that in the propos'd Equation ſeveral flowing Quantities may be involved; and that the Terms may be multiply'd, not only by the Number of the Dimenſions of the flowing Quantities, but alſo by any other Arithmetical Progreſſions; ſo that in the Operation there may be the ſame difference of the Terms according to any of the flowing Quantities, and the Progreſſion be diſpos'd according to the ſame order of the Dimenſions of each of them. And theſe things being allow'd, what is taught beſides in Examp. 3, 4, and 5, will be plain enough of itſelf.

PROB. II.

An Equation being propoſed, including the Fluxions of Quantities, to find the Relations of thoſe Quantities to one another.

A PARTICULAR SOLUTION.

1. As this Problem is the Converſe of the foregoing, it muſt be ſolved by proceeding in a contrary manner. That is, the Terms multiply'd by $\dot{x}$ being diſpoſed according to the Dimenſions of x; they muſt be divided by $\frac{\dot{x}}{x}$, and then by the number of their Dimenſions, or perhaps by ſome other Arithmetical Progreſſion. Then the ſame work muſt be repeated with the Terms multiply'd by $\dot{v}$, $\dot{y}$,

Newton's calculus in action (*The method of fluxions*, 1736)

12. Ex. 5. As if the Equation $zz + axz - y^4 = 0$ were propos'd to expreſs the Relation between x and y, as alſo $\sqrt{ax - xx} = BD$, for determining a Curve, which therefore will be a Circle. The Equation $zz + axz - y^4 = 0$, as before, will give $2\dot{z}z + a\dot{z}x + ax\dot{z} - 4\dot{y}y^3 = 0$, for the Relation of the Celerities $\dot{x}$, $\dot{y}$, and $\dot{z}$. And therefore ſince it is $z = x \times BD$ or $= x\sqrt{ax - xx}$, ſubſtitute this Value inſtead of it, and there will ariſe the Equation $2\dot{x}z + a\dot{x}x\sqrt{ax - xx} + ax\dot{z} - 4\dot{y}y^3 = 0$, which determines the Relation of the Celerities $\dot{x}$ and $\dot{y}$.

Demonstration *of the Solution.*

13. The Moments of flowing Quantities, (that is, their indefinitely ſmall Parts, by the acceſſion of which, in indefinitely ſmall portions of Time, they are continually increaſed,) are as the Velocities of their Flowing or Increaſing.

14. Wherefore if the Moment of any one, as x, be repreſented by the Product of its Celerity $\dot{x}$ into an indefinitely ſmall Quantity o (that is, by $\dot{x}o$,) the Moments of the others v, y, z, will be repreſented by $\dot{v}o$, $\dot{y}o$, $\dot{z}o$; becauſe $\dot{v}o$, $\dot{x}o$, $\dot{y}o$, and $\dot{z}o$, are to each other as $\dot{v}$, $\dot{x}$, $\dot{y}$, and $\dot{z}$.

15. Now ſince the Moments, as $\dot{x}o$ and $\dot{y}o$, are the indefinitely little acceſſions of the flowing Quantities x and y, by which thoſe Quantities are increaſed through the ſeveral indefinitely little intervals of Time; it follows, that thoſe Quantities x and y, after any indefinitely ſmall interval of Time, become $x + \dot{x}o$ and $y + \dot{y}o$. And therefore the Equation, which at all times indifferently expreſſes the Relation of the flowing Quantities, will as well expreſs the Relation between $x + \dot{x}o$ and $y + \dot{y}o$, as between x and y: So that $x + \dot{x}o$ and $y + \dot{y}o$ may be ſubſtituted in the ſame Equation for thoſe Quantities, inſtead of x and y.

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$$\left.\begin{array}{llll} x^3 & +3\dot{x}ox^2 & +3\dot{x}^2o^2x & +\dot{x}^3o^3 \\ -ax^2 & -2a\dot{x}ox & -a\dot{x}^2o^2 & \\ +axy & +a\dot{x}oy & +ay\dot{x}o & +a\dot{x}yo^2 \\ -y^3 & -3\dot{y}oy^2 & -3\dot{y}^2o^2y & -\dot{y}^3o^3 \end{array}\right\} = 0.$$

17. Now by Suppoſition $x^3 - ax^2 + axy - y^3 = 0$, which therefore being expunged, and the remaining Terms being divided by o, there will remain $3\dot{x}x^2 + 3\dot{x}^2ox + \dot{x}^3o^2 - 2a\dot{x}x - a\dot{x}^2o + a\dot{x}y + ay\dot{x} + a\dot{x}yo - 3\dot{y}y^2 - 3\dot{y}^2oy - \dot{y}^3o^2 = 0$. But whereas o is ſuppoſed to be infinitely little, that it may repreſent the Moments of Quantities; the Terms that are multiply'd by it will be nothing in reſpect of the reſt. Therefore I reject them, and there remains $3\dot{x}x^2 - 2a\dot{x}x + a\dot{x}y + ay\dot{x} - 3\dot{y}y^2 = 0$, as above in Examp. 1.

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PROB. II.

An Equation being propoſed, including the Fluxions of Quantities, to find the Relations of thoſe Quantities to one another.

A Particular Solution.

1. As this Problem is the Converſe of the foregoing, it muſt be ſolved by proceeding in a contrary manner. That is, the Terms multiply'd by $\dot{x}$ being diſpoſed according to the Dimenſions of x; they muſt be divided by $\frac{\dot{x}}{x}$, and then by the number of their Dimenſions, or perhaps by ſome other Arithmetical Progreſſion. Then the ſame work muſt be repeated with the Terms multiply'd by $\dot{v}$, $\dot{y}$,

Leibniz's calculus

Independently, ten years later than Newton...

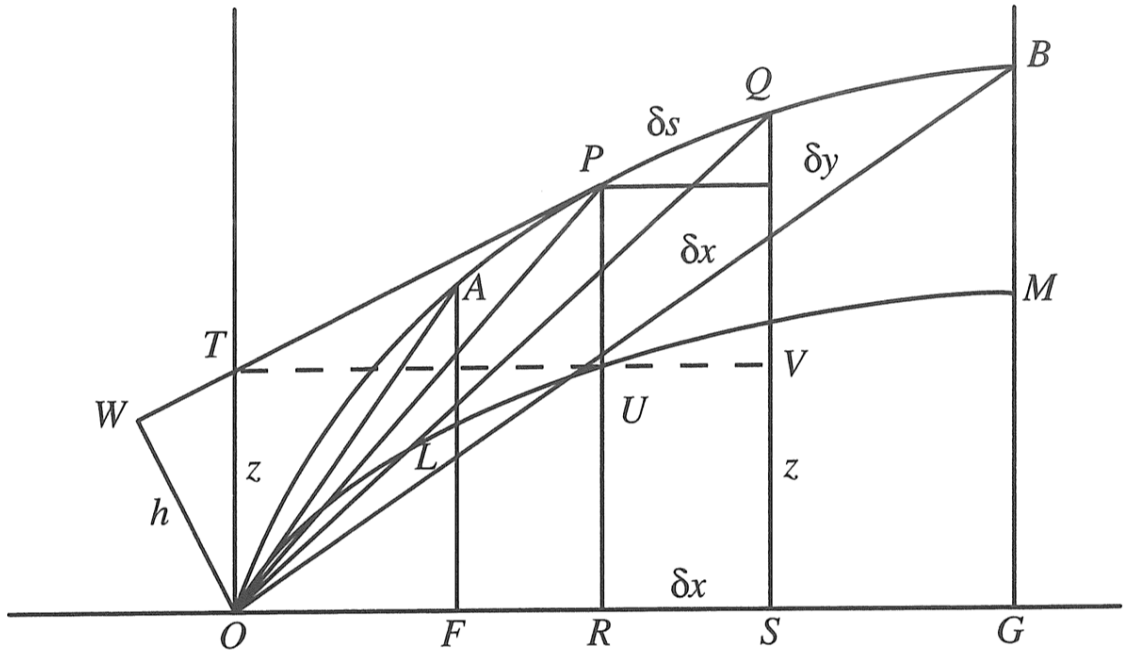
Leibniz's calculus, 1673–76:

- rules for quadrature — especially the transformation theorem (a.k.a. the transmutation theorem)
- rules for tangents — by characteristic (or differential) triangles
- recognition that these are inverse processes

Differentials: du, dv;

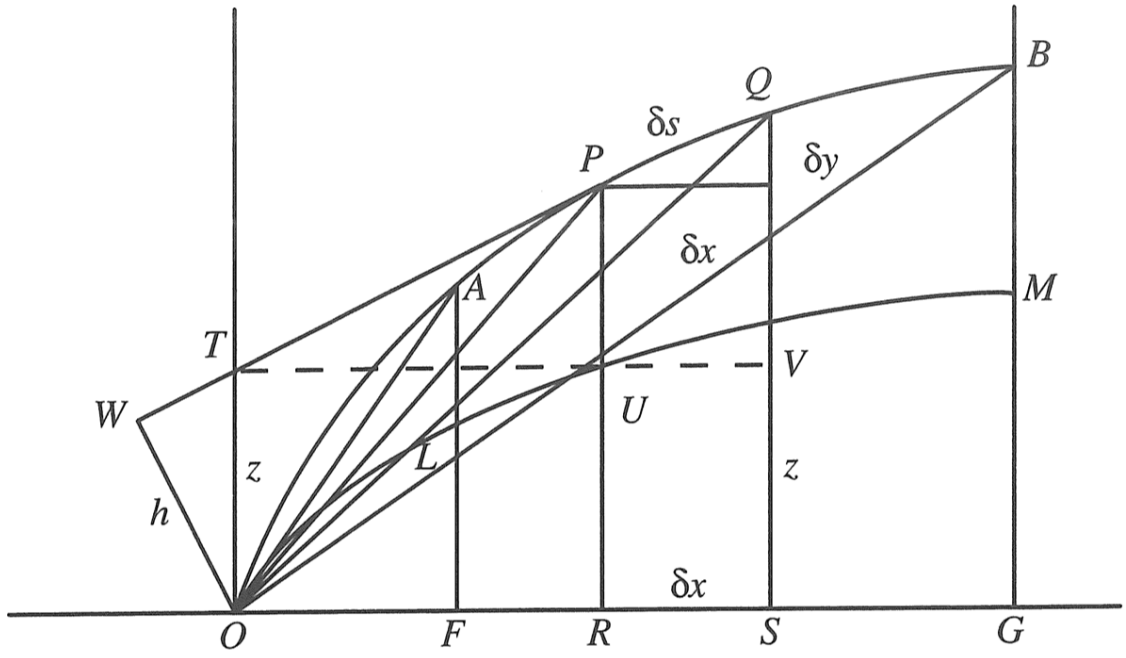
integrals: omn. l, later between $S\mathrm{l}$ and $\int \mathrm{l}$

Leibniz's Transmutation Theorem



$$OABG = \sum RPQS = \frac{1}{2} OG \cdot GB + \sum OPQ$$

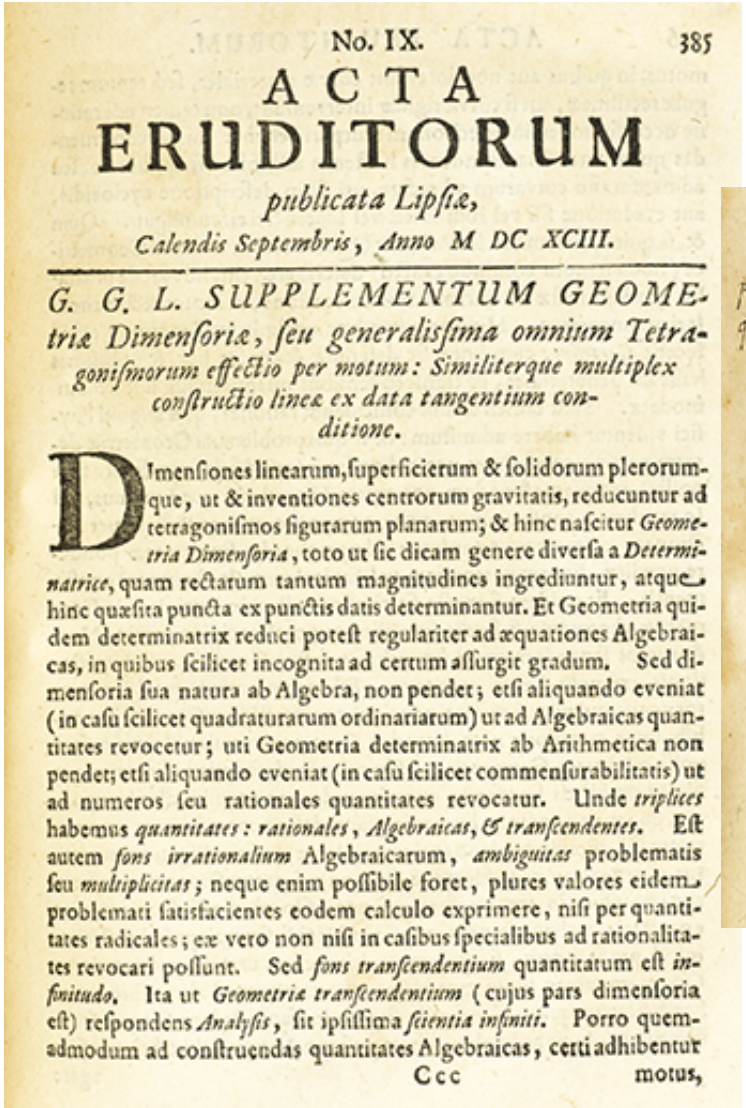
Leibniz's Transmutation Theorem



$$OABG = \sum RPQS = \frac{1}{2} OG \cdot GB + \sum OPQ$$

$$= \frac{1}{2} OG \cdot GB + \frac{1}{2} OLMG$$

Supplementum Geometriae Dimensoriae . . . (1693)



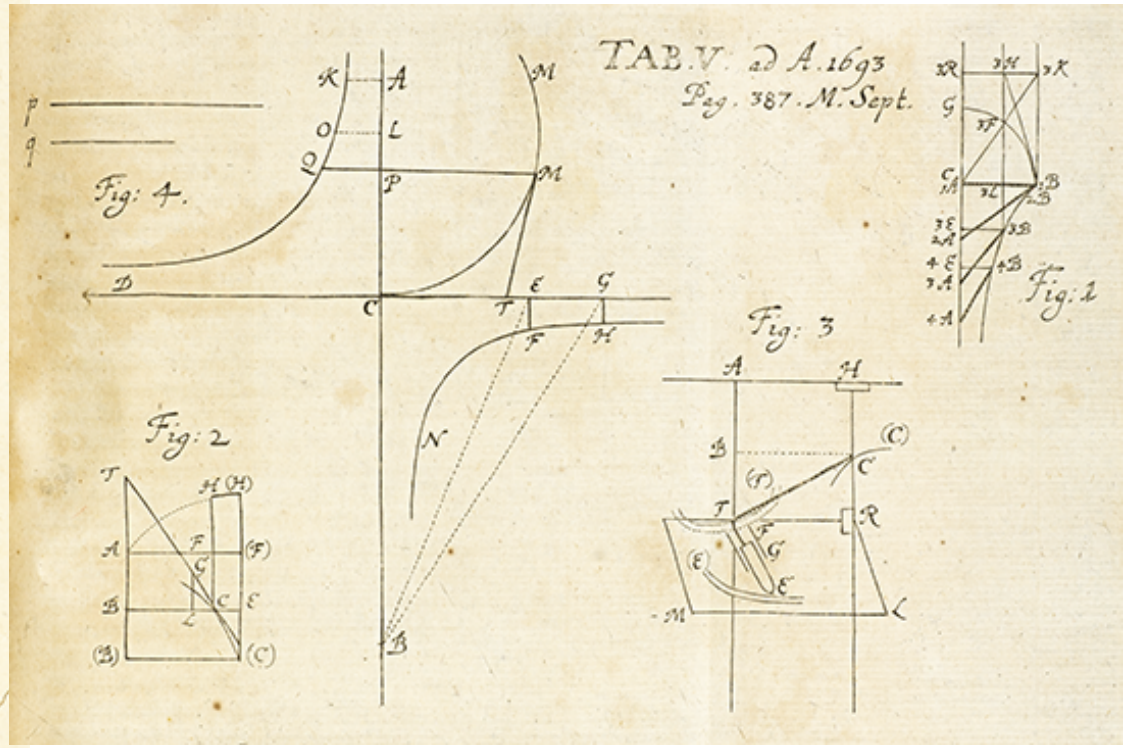
No. IX. 385

ACTA
ERUDITORUM
publicata Lipſiæ,
Calendis Septembris, Anno M DC XCIII.

G. G. L. SUPPLEMENTUM GEOMETRIÆ Dimenſoriæ, ſeu generaliſſima omnium Tetragoniſmorum effectio per motum: Similiterque multiplex conſtructio lineæ ex data tangentium conditione.

Dimenſiones linearum, ſuperficierum & ſolidorum plerorumque, ut & inventiones centrorum gravitatis, reducuntur ad tetragoniſmos figurarum planarum; & hinc naſcitur *Geometria Dimenſoria*, toto ut ſic dicam genere diverſa a *Determinatrice*, quam rectarum tantum magnitudines ingrediuntur, atque hinc quæſita puncta ex punctis datis determinantur. Et Geometria quidem determinatrix reduci poteſt regulariter ad æquationes Algebraicas, in quibus ſcilicet incognita ad certum aſſurgit gradum. Sed dimenſoria ſua natura ab Algebra, non pendet; etſi aliquando eveniat (in caſu ſcilicet quadraturarum ordinariarum) ut ad Algebraicas quantitates revocetur; uti Geometria determinatrix ab Arithmetica non pendet; etſi aliquando eveniat (in caſu ſcilicet commenſurabilitatis) ut ad numeros ſeu rationales quantitates revocatur. Unde *triplices* habemus *quantitates: rationales, Algebraicas, & tranſcendentes.* Eſt autem *fons irrationalium* Algebraicarum, *ambiguitas* problematis ſeu *multiplicitas*; neque enim poſſibile foret, plures valores eidem problemati ſatisfacientes eodem calculo exprimere, niſi per quantitates radicales; ex vero non niſi in caſibus ſpecialibus ad rationalitates revocari poſſunt. Sed *fons tranſcendentium* quantitatum eſt *infinitudo*. Ita ut *Geometria tranſcendentium* (cujus pars dimenſoria eſt) reſpondens *Analyſis*, ſit ipſiſſima *ſcientia infiniti*. Porro quemadmodum ad conſtruendas quantitates Algebraicas, certi adhibentur

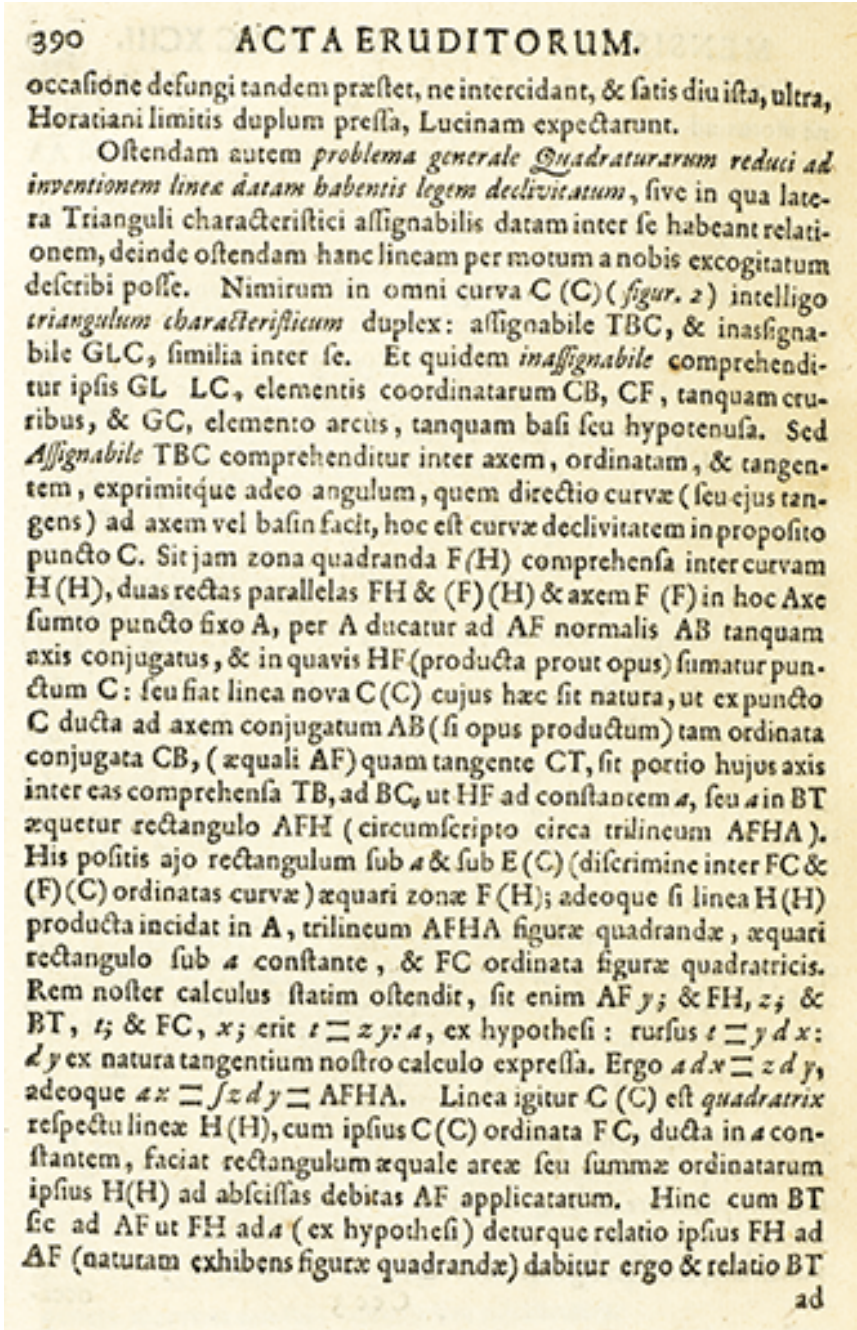
Ccc motus,



A Supplement to the Geometry of Measurements, or the Most General of all Quadratures to be Effected by a Motion: and likewise the various constructions of a curve from a given condition of the tangent

(*Acta eruditorum*, 1693)

Supplementum Geometriae Dimensoriae . . . (1693)



390 ACTA ERUDITORUM.

occasione defungi tandem præstet, ne intercidant, & satis diu ista, ultra, Horatiani limitis duplum pressa, Lucinam expectarunt.

Ostendam autem *problema generale Quadraturarum reduci ad inventionem lineæ datam habentis legem declivitatum*, sive in qua latera Trianguli characteristici assignabilis datam inter se habeant relationem, deinde ostendam hanc lineam per motum a nobis excogitatum describi posse. Nimirum in omni curva C (C) (*figur. 2*) intelligo *triangulum characteristicum* duplex: assignabile TBC, & inassignabile GLC, similia inter se. Et quidem *inassignabile* comprehenditur ipsis GL LC, elementis coordinatarum CB, CF, tanquam cruribus, & GC, elemento arcus, tanquam basi seu hypotenusa. Sed *Assignabile* TBC comprehenditur inter axem, ordinatam, & tangentem, exprimitque adeo angulum, quem directio curvæ (seu ejus tangens) ad axem vel basin facit, hoc est curvæ declivitatem in proposito puncto C. Sit jam zona quadranda F(H) comprehensa inter curvam H (H), duas rectas parallelas FH & (F) (H) & axem F (F) in hoc Axe sumto puncto fixo A, per A ducatur ad AF normalis AB tanquam axis conjugatus, & in quavis HF (producta prout opus) sumatur punctum C: seu fiat linea nova C (C) cujus hæc sit natura, ut ex puncto C ducta ad axem conjugatum AB (si opus productum) tam ordinata conjugata CB, (æquali AF) quam tangente CT, sit portio hujus axis inter eas comprehensa TB, ad BC, ut HF ad constantem *a*, seu *a* in BT æquetur rectangulo AFH (circumscripto circa trilineum AFHA). His positis ajo rectangulum sub *a* & sub E (C) (discrimine inter FC & (F) (C) ordinatas curvæ) æquari zonæ F (H); adeoque si linea H (H) producta incidat in A, trilineum AFHA figuræ quadrandæ, æquari rectangulo sub *a* constante, & FC ordinata figuræ quadratricis. Rem noster calculus statim ostendit, sit enim AF y; & FH, z; & BT, t; & FC, x; erit $t = z\,y : a$, ex hypothesi: rursus $t = y\,dx : dy$ ex natura tangentium nostro calculo expressa. Ergo $a\,dx = z\,dy$, adeoque $ax = \int z\,dy =$ AFHA. Linea igitur C (C) est *quadratrix* respectu lineæ H (H), cum ipsius C (C) ordinata FC, ducta in *a* constantem, faciat rectangulum æquale areæ seu summæ ordinatarum ipsius H (H) ad abscissas debitas AF applicatarum. Hinc cum BT sit ad AF ut FH ad *a* (ex hypothesi) deturque relatio ipsius FH ad AF (naturam exhibens figuræ quadrandæ) dabitur ergo & relatio BT

ad

"I shall now show *the general problem of quadratures to be reduced to the invention of a line having a given law of declivity*"

i.e., integration is reduced to the finding of a curve with a particular tangent — in modern terms, the antiderivative

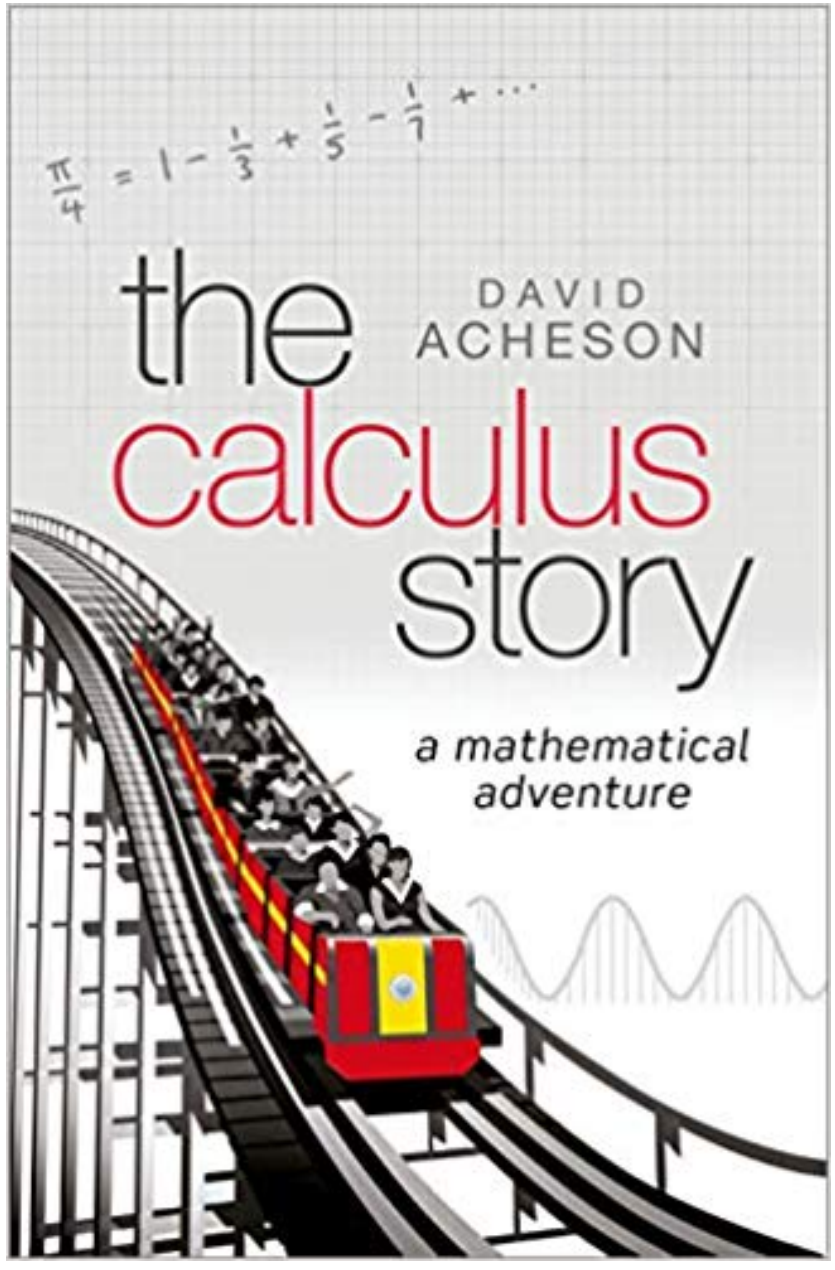
For Latin-readers: full paper available online

See also: Niccolò Guicciardini, Newton's method and Leibniz's calculus, *A history of analysis* (ed. Hans Niels Jahnke), AMS/LMS, 2003, pp. 73–103

Newton's calculus and Leibniz's calculus compared

Newton (1664–65):	Leibniz (1673–76):
rules for quadrature	rules for quadrature
rules for tangents	rules for tangents
'fundamental theorem'	'fundamental theorem'
dot notation	'modern' notation
physical intuition:	algebraic intuition
rates of change	rules and procedures
PROBLEM:	PROBLEM:
vanishing quantities o	vanishing quantities du, dv, ...

An elementary introduction to the development of calculus

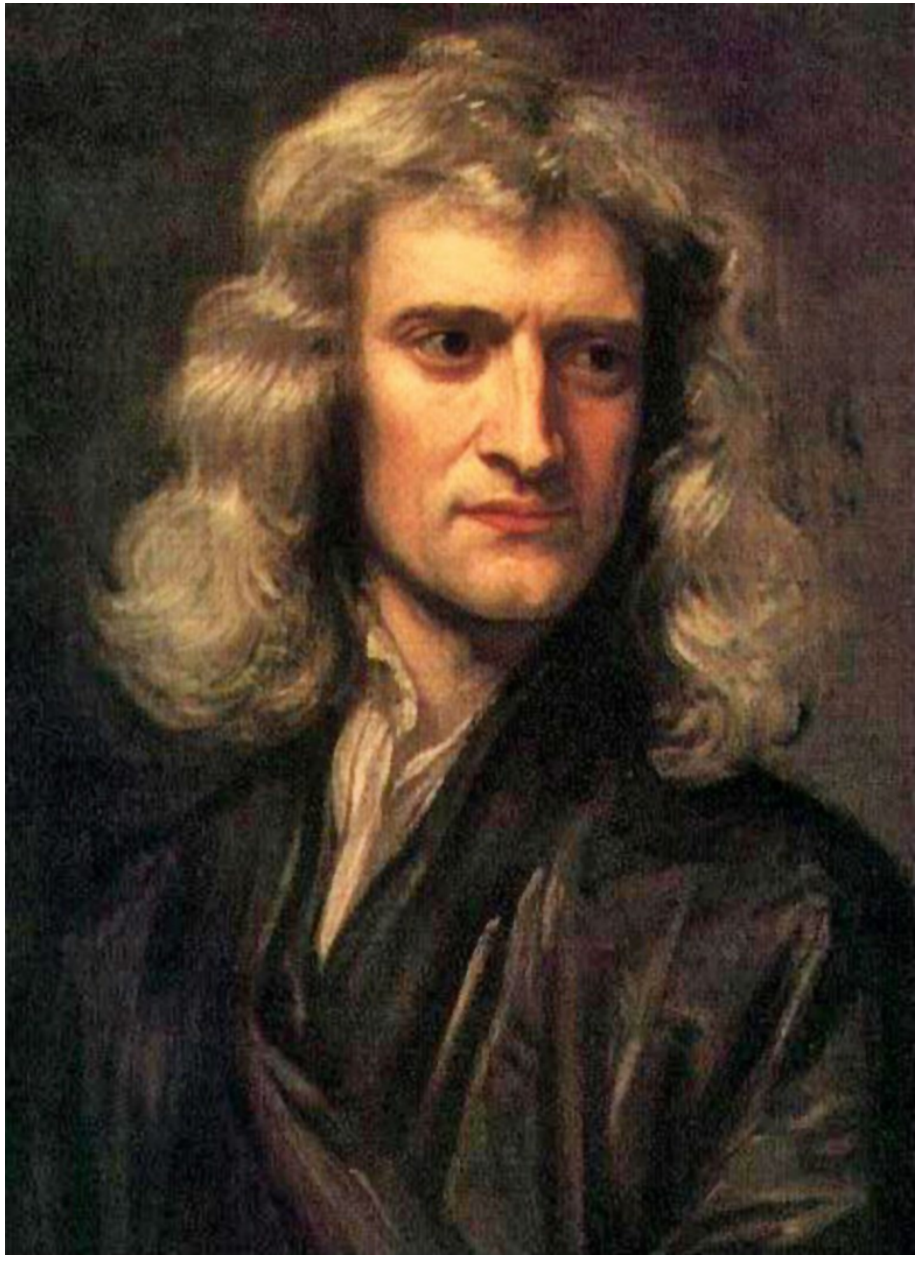


BO1 History of Mathematics
Lecture V
Newton's *Principia*

MT 2022 Week 3

Summary

- Isaac Newton (1642–1727)
- Kepler's laws, Descartes' theory, Hooke's conjecture
- The *Principia*
- Editions and influence of the *Principia*



Newton summarised

Alexander Pope, 1730:

> *Nature and Nature's Laws lay hid in Night.*
> *God said, Let Newton be! and All was Light.*

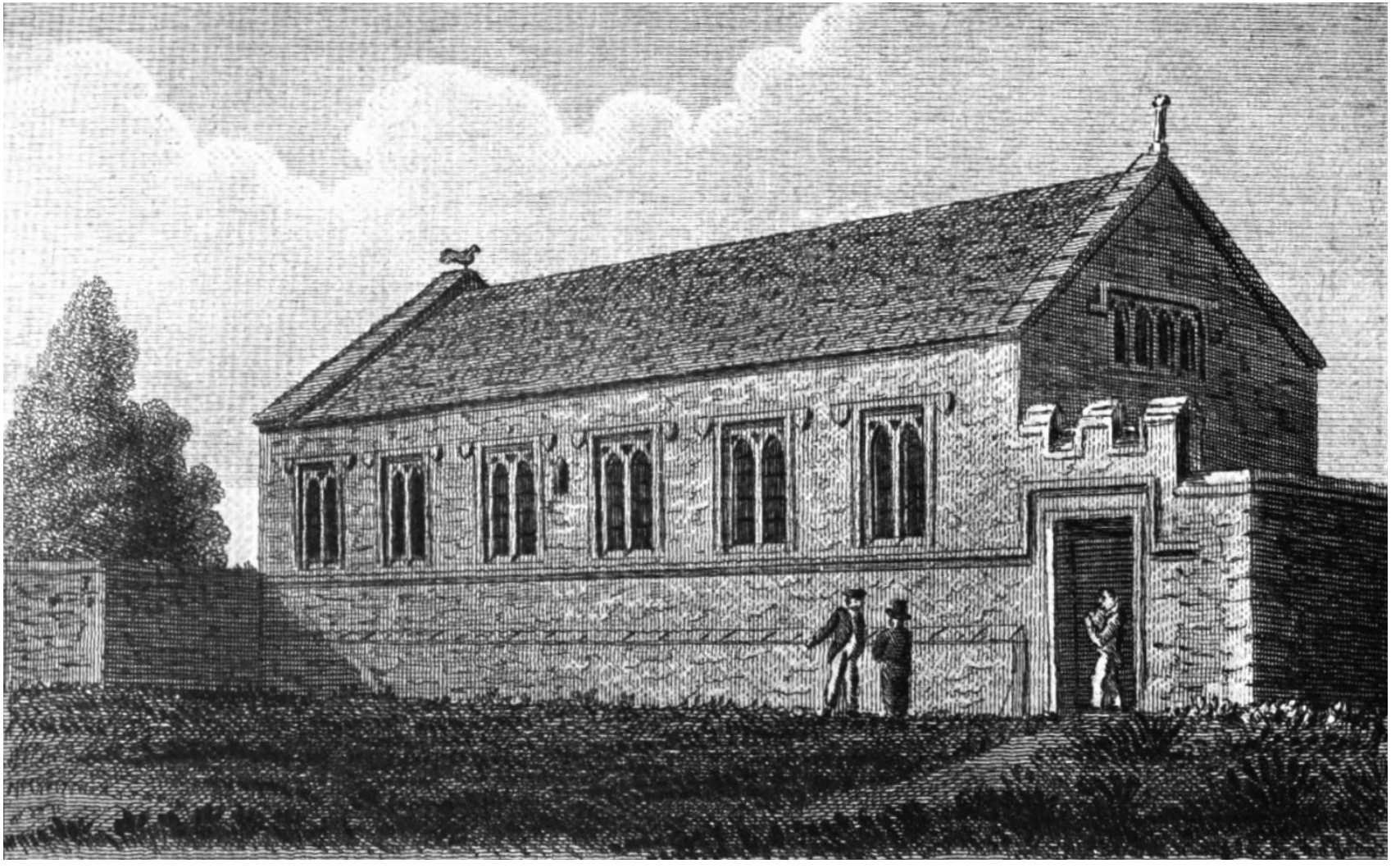
Woolsthorpe Manor

Engraved for the History of Lincolnshire
WOLSTHORPE.
THE HOUSE IN WHICH SIR ISAAC NEWTON WAS BORN
To C. Babbage Esq^r this Plate is respectfully inscribed
by the Author.
LONDON. PUBLISHED 1832.

Newton born at Woolsthorpe Manor, 25th December 1642

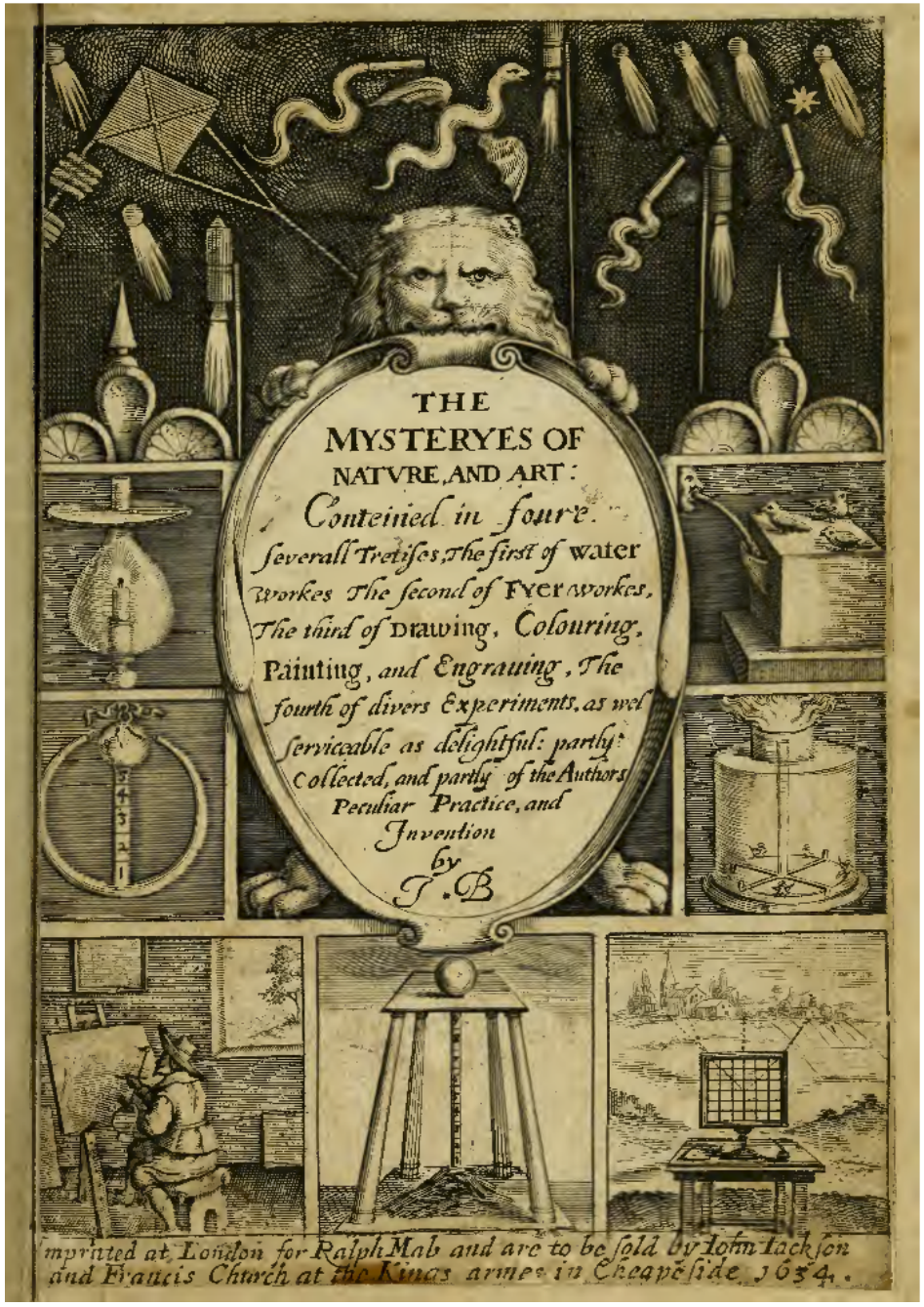
Woolsthorpe Manor

Grantham Grammar School



Now The King's School, Grantham

John Bate: *The mysteries of nature and art* (1634)


THE
MYSTERYES OF
NATVRE, AND ART:
Conteined in foure
severall Tretises, The first of water
Workes The second of Fyer workes,
The third of Drawing, Colouring,
Painting, and Engraving, The
fourth of divers Experiments, as wel
serviceable as delightful: partly
Collected, and partly of the Authors
Peculiar Practice, and
Invention
by
I. B
Imprinted at London for Ralph Mab and are to be sold by Iohn Iackson
and Francis Church at the Kings armes in Cheapeside. 1634.

John Bate: *The mysteries of nature and art* (1634)

little from it on the ſide. Having thus prepared the barrell, fit a good thick board unto it, ſo that it may ſlip eaſily up and down from the top of the barrell unto the bottom, nayle a lether about the edges of it, and another upon the top of it: on the underſide of it let there be faſtned a good ſtiffe, but flexible ſpring of ſteele, which may thruſt the board from the bottom to the top of the barrell: let the foot of this ſpring reſt upon a barre faſtned acros the bottom of the barrell; let this board alſo have tied at the middle a little rope of length ſufficient. When you uſe it, bore a little hole in the table that you ſet it on, to put the rope thorow, and pull the rope down, which will contract the ſpring, and with it draw down the board: then poure in water at the baſin untill the veſſell be full: Note then, as you let ſlack the rope, the water will ſpirt out of the pipe, in the middle, and as you pull it ſtraight, the water will run into the veſſell againe. You may make birds, or divers images at the top of the pipe, out of which the water may break.

Another

Another manner of forcing water, whereby the water of any ſpring may be forced unto the top of a hill.

LEt there be two hollow poſts, with a ſuccur at the bottom of each, alſo a ſuccur nigh the top of each: let there be faſtned unto both theſe poſts a ſtrong peece of

timber, having, as it were, a beame or ſcale pinned in it, and having two handles, at each end one. In the tops of

John Bate: *The mysteries of nature and art* (1634)

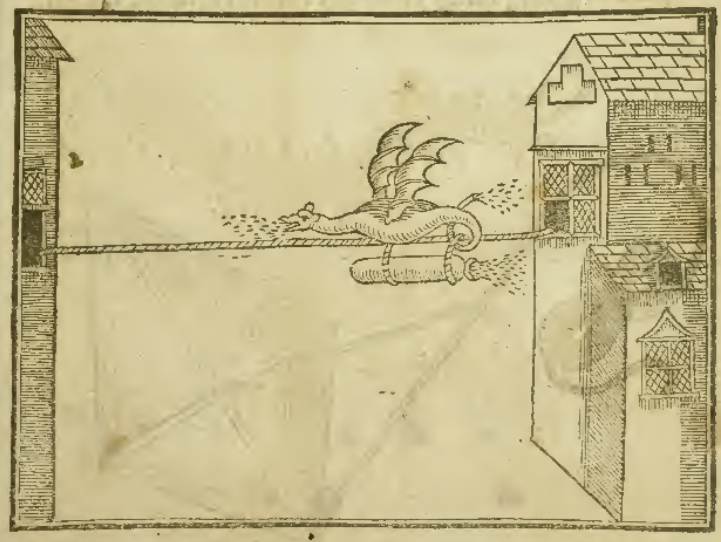
THE
SECOND BOOKE,
Teaching moſt plainly, and withall
moſt exactly, the compoſing of all
manner of Fire-works for Triumph
and Recreation.

By *I. B.*

LONDON,
Printed by *Thomas Harper* for *Ralph Mab.* 1634.

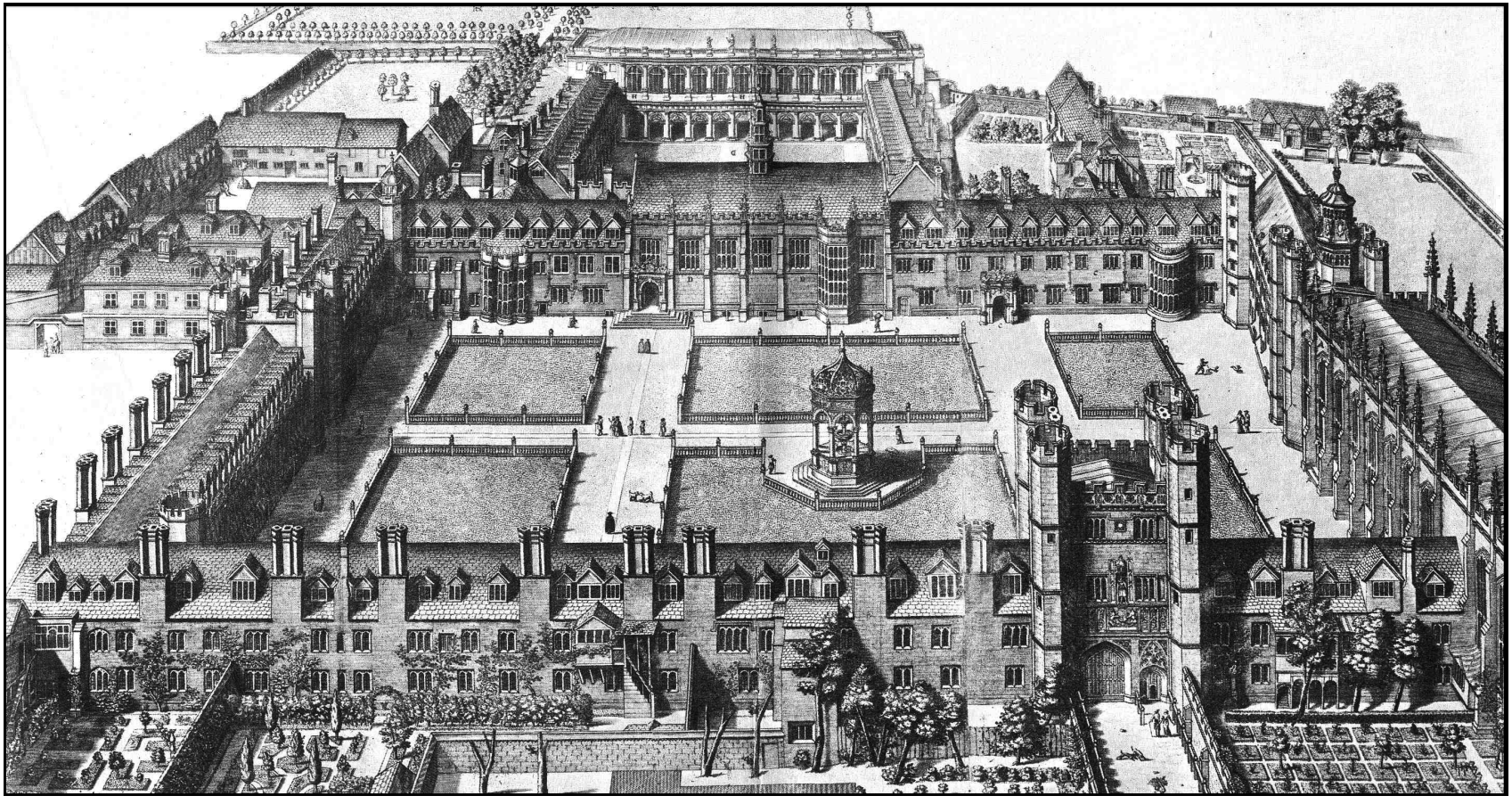
How to make flying Dragons.

THe flying Dragon is ſomewhat troubleſome to compoſe; it muſt be made eyther of dry and light wood, or crooked-lane plates, or of thin whalebones covered with Muſcovie glaſſe, and painted over. In the body thereof, there muſt bee a voyde cane to paſſe the rope through; unto the bottome of this cane muſt bee bound one or two large Rockets, according as the bigneſſe and



weight of the Dragon ſhall require; the body muſt bee filled with divers petrars, that may conſume it, and a ſparkling receipt muſt be ſo diſpoſed upon it, that being fired, it may burne both at the mouth and at the tayle thereof;

Trinity College, Cambridge



Isaac Newton (1642–1727)

Newton's major interests:

1650s:	model-building
1660s:	optics; (pure) mathematics
1670s:	alchemy, theology
1684+:	mathematics
1696–1727:	Warden of the Mint

Johannes Kepler (1571–1630)

Engaged to sift through the astronomical data gathered by the Danish astronomer Tycho Brahe (1546–1601)

Major works:

Astronomia nova (1609)

Harmonices mundi (1619)

Kepler: *Astronomia nova* (1609)

ASTRONOMIA NOVA
AITIOΛOΓHTOΣ,
SEV
PHYSICA COELESTIS,
tradita commentariis
DE MOTIBVS STELLÆ
MARTIS,
Ex obſervationibus G. V.
TYCHONIS BRAHE:

Juſſu & ſumptibus
RVDOLPHI II.
ROMANORVM
IMPERATORIS &c:

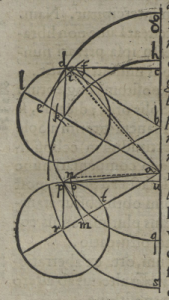
Plurium annorum pertinaci ſtudio
elaborata Pragæ,

A Sa. Ca. M.tis Sa. Mathematico
JOANNE KEPLERO,

Cum ejusdem Ca. M.tis privilegio ſpeciali
ANNO ætæ Dionyſianæ clɔ Iɔ c ix.

CAP. LVIII.

ſumus mutuari: idem per hanc etiam veriſſimam hypotheſin tentans, falſa methodo, rurſum de rerum ſumma trepidare cœpi. *In linea apſidum; centris* A. B. *ſcribantur æquales circuli* G D. H K. *Sitque* A B *eccentricitas circuli* G D. *Sit autem anomalia eccentri, ſeu numerus graduum ejus, arcus* G D *vel* H K, *per æquipollentiam capitis* I I I. *Centro igitur* K, *diaſtemate* K D, *quod ipſi* A B *ſit æquale, ſcribatur* L D F *epicyclus, qui ſecabit circulum* G D *in* D, *per æquipollentiam cap.* I I I. *Ducatur* A K, *& continuetur donec ſecet epicyclum in* L, *ut ſit* L D *arcus ſimilis anomaliæ eccentri* G D *vel* H K. *Et connectatur* B *cum* D. *Ex puncto vero* D *demittantur perpendiculares in* G A, L A, *quæ ſint* D C, D E.



Quare per hactenus cap. L V I. demonſtrata, A E citra controverſiam erit juſta diſtantia ad hanc anomaliam eccentri, de qua quæritur, quantum temporis in ea ſit conſumptum. Cumque ejus arcus ſinus verſus G C, ſive poſt multiplicationem, L E ablatâ A G A, prodiderit diſtantiam A E juſtam: ex his indiciis perſuadebar, terminum ipſius A E alterum, quærendum eſſe non in D C linea, quod veriſſimum tamen erat, ſed in D B lineæ puncto I: ut ſi centro A, diaſtemate A E, ducerem arcum E I F, qui ſecet D B in I. Eſſet igitur A I ſecundum hanc perſuaſionem juſta diſtantia, ſitu & longitudine; & I A G anomalia vere coæquata. Manifeſtum eſt autem, quod E I F arcus ſecet D C lineam, loco ſuperiori ſcilicet in F. itaque anguli I A G & F A G differant quantitate I A F.

Erravi igitur, uſurpata linea A I pro A F. Errorem primum experientia deprehendi. Nam cum exploraſſem quantitatem areæ D A G tam per diſtantias omnes quam per areolam D A B, poſtea huic areæ D A G in tempus converſæ accommodaſſem angulum I A G non F A G; tunc in ſuperiori ſemicirculi parte collegi per $\frac{5}{4}$ plus, in inferiori per $\frac{4}{5}$ minus, quam dabat vicaria, ſatis certa. Itaque diſſentientibus æquationibus a vero, cœpi rurſum accuſare veriſſimas has diſtantias A E, & librationem Planetæ L E, de crimine, cujus falſa mea methodus, quæ I pro F ſpectabat, erat rea. Quid multis? Ipſa veritas & rerum Natura repudiata, & exulare juſſa, per poſticum ſe furtim rurſum recepit intro, & ſub habitu alieno a me recepta fuit. Miſſis inquam librationibus diametri L E, cœpi revocare ellipſes, omnino exiſtimans, me ſic longe diverſiſſimam, a librationibus, ſequi hypotheſin; cum plane coincidant, ut cap. ſequente demonſtrabitur: niſi quod, quæ peccaveram prius in methodo, hac ratione fuerunt emendata, & F pro I, ita ut debuit, uſurpatum.

Argumentatio mea talis fuit, qualis cap. X L I X. L. & L V I. Circulus cap. X L I I I. peccat exceſſu, ellipſis capitis X L V peccat defectu. Et ſunt exceſſus ille & hic defectus æquales. Inter circulum vero & ellipſin

nihil

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Fol. Ast. 30-3 2 copy

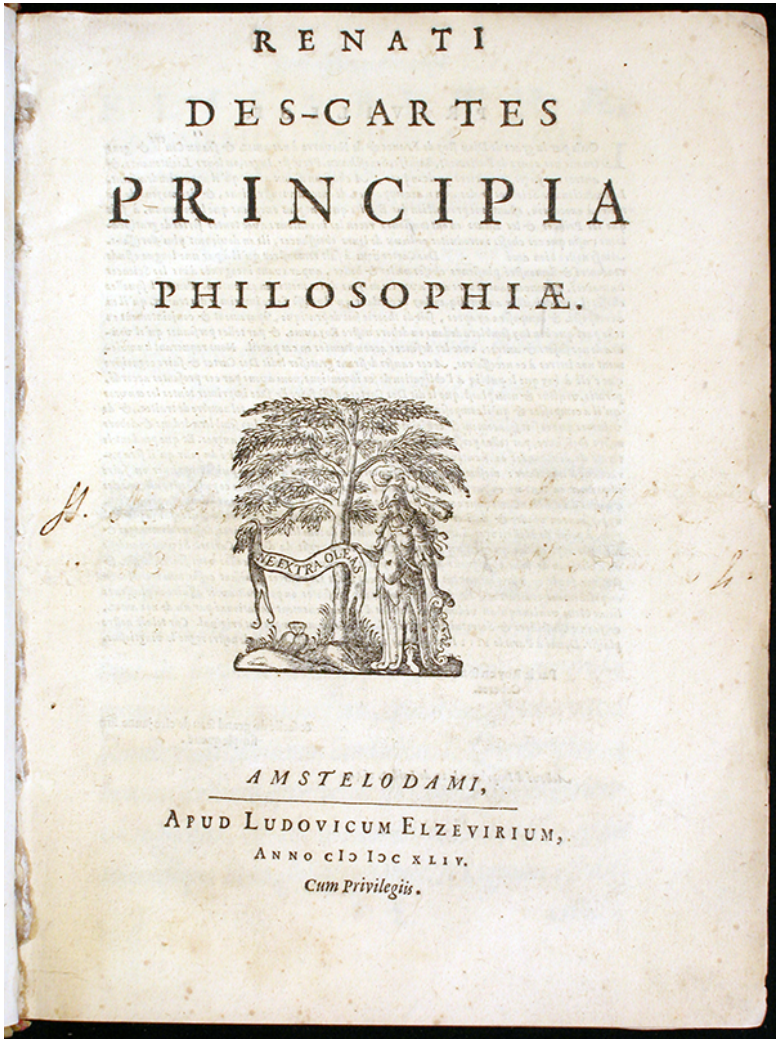
Kepler's laws

Kepler's laws of planetary motion (1609, 1619):

1. Planets move in elliptical orbits with the sun as focus
2. Planets sweep out equal areas in equal times
3. T^2 is proportional to R^3 (where T is time of one revolution, R is mean distance to sun)

All from empirical evidence

Descartes' theory


RENATI

DES-CARTES

PRINCIPIA

PHILOSOPHIÆ.

NE EXTRA OLEA

AMSTELODAMI,

APUD LUDOVICUM ELZEVIRIUM,

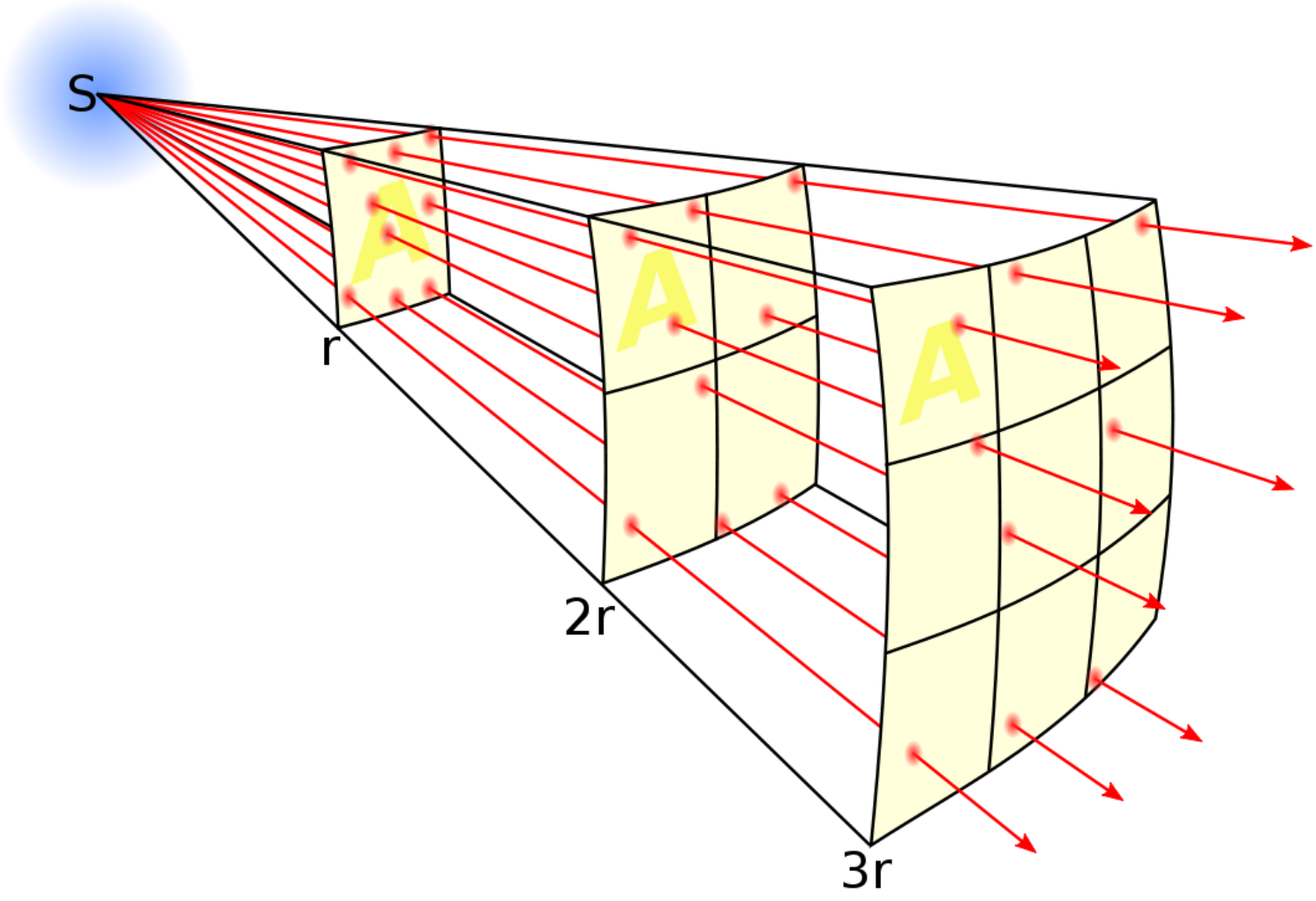
ANNO CIↃ IↃC XLIV.

Cum Privilegiis.

Descartes' views of planetary motion in *Principia philosophiae* (1644):

- the sun is one star among many
- asserted that planets are carried round their suns by vortices of the surrounding 'ether'
- claimed that theory could also explain magnetism and static electricity

An inverse square law?

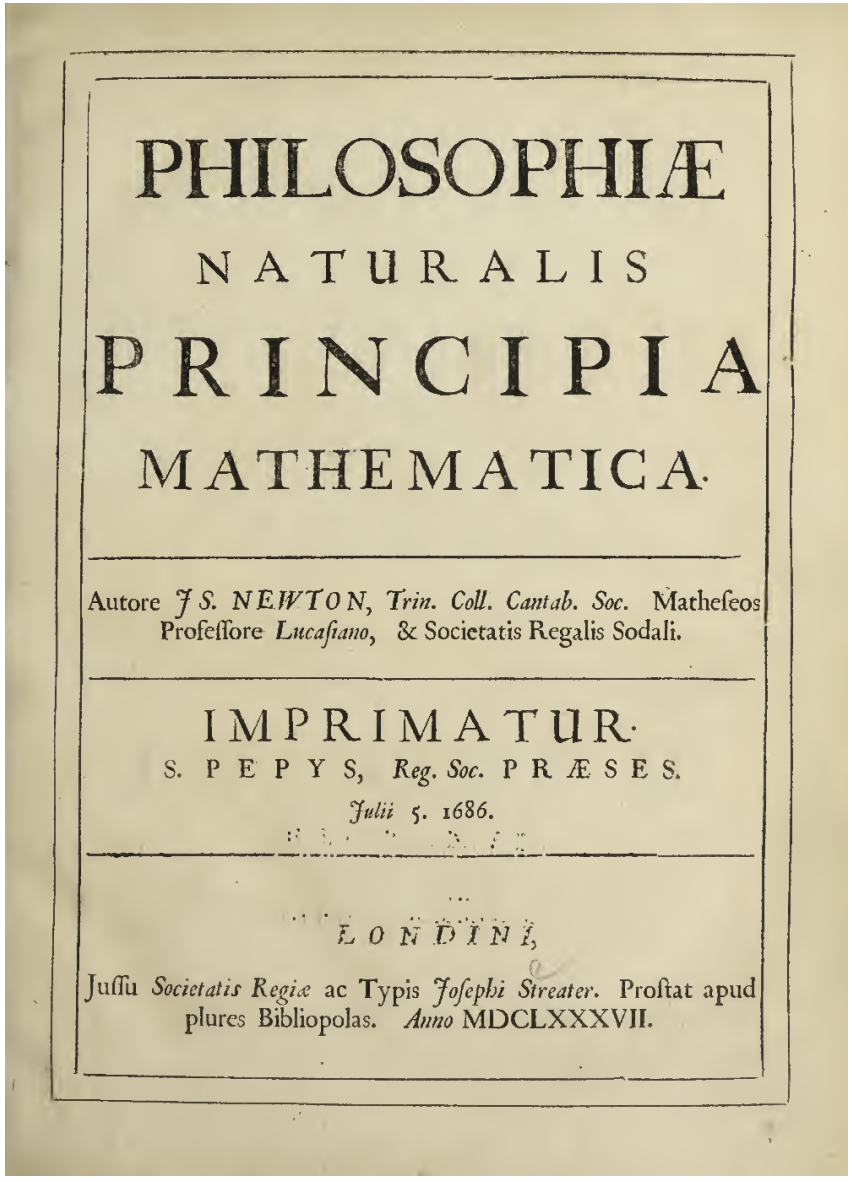


An inverse square law?

Speculations and calculations on an inverse square law of gravity:

1645	Ismaël Bullialdus refutes a claim of Kepler that 'gravity' drops off linearly with distance, instead suggesting an inverse square law
c. 1679	Hooke corresponds with Newton and suggests that an inverse square law might lead to elliptical orbits
1684	Halley visits Newton and asks whether this might be the case; Newton sends a short treatise on motion to Halley
1687	Publication of Newton's *Principia* at Halley's expense

Isaac Newton: *The mathematical principles of natural philosophy* (London, 1687)



PHILOSOPHIÆ
NATURALIS
PRINCIPIA
MATHEMATICA.

Autore *J S. NEWTON, Trin. Coll. Cantab. Soc.* Matheſeos Profeſſore *Lucaſiano*, & Societatis Regalis Sodali.

IMPRIMATUR.
S. PEPYS, *Reg. Soc.* PRÆSES.
Julii 5. 1686.

LONDINI,
Juſſu *Societatis Regiæ* ac Typis *Joſephi Streater*. Proſtat apud plures Bibliopolas. *Anno* MDCLXXXVII.

Contents of the *Principia*

- Eight definitions — of matter, motion, innate force, impressed force, acceleration, ...
- Three axioms or Laws of Motion (as taught in school), together with six corollaries
- Book I: The motion of bodies
- Book II: The motion of bodies in resisting media
- Book III: The system of the world

The laws of motion

AXIOMATA
SIVE
LEGES MOTUS

Lex. I.

Corpus omne perſeverare in ſtatu ſuo quieſcendi vel movendi uniformiter in directum, niſi quatenus a viribus impreſſis cogitur ſtatum illum mutare.

PRojectilia perſeverant in motibus ſuis niſi quatenus a reſiſtentia aeris retardantur & vi gravitatis impelluntur deorſum. Trochus, cujus partes cohærendo perpetuo retrahunt ſeſe a motibus rectilineis, non ceſſat rotari niſi quatenus ab aere retardatur. Majora autem Planetarum & Cometarum corpora motus ſuos & progreſſivos & circulares in ſpatiis minus reſiſtentibus factos conſervant diutius.

Lex. II.

Mutationem motus proportionalem eſſe vi motrici impreſſæ, & fieri ſecundum lineam rectam qua vis illa imprimitur.

Si vis aliqua motum quemvis generet, dupla duplum, tripla triplum generabit, ſive ſimul & ſemel, ſive gradatim & ſucceſſive impreſſa fuerit. Et hic motus quoniam in eandem ſemper plagam cum vi generatrice determinatur, ſi corpus antea movebatur, motui ejus vel conſpiranti additur, vel contrario ſubducitur, vel obliquo oblique adjicitur, & cum eo ſecundum utriuſq; determinationem componitur. Lex. III.

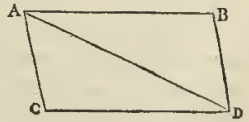
Lex. III.

Actioni contrariam ſemper & æqualem eſſe reactionem: ſive corporum duorum actiones in ſe mutuo ſemper eſſe æquales & in partes contrarias dirigi.

Quicquid premit vel trahit alterum, tantundem ab eo premitur vel trahitur. Siquis lapidem digito premit, premitur & hujus digitus a lapide. Si equus lapidem funi allegatum trahit, retrahetur etiam & equus æqualiter in lapidem: nam funis utrinq; diſtentus eodem relaxandi ſe conatu urgebit Equum verſus lapidem, ac lapidem verſus equum, tantumq; impediet progreſſum unius quantum promovet progreſſum alterius. Si corpus aliquod in corpus aliud impingens, motum ejus vi ſua quomodocunq: mutaverit, idem quoque viciſſim in motu proprio eandem mutationem in partem contrariam vi alterius (ob æqualitatem preſſionis mutuæ) ſubibit. His actionibus æquales fiunt mutationes non velocitatum ſed motuum, (ſcilicet in corporibus non aliunde impeditis:) Mutationes enim velocitatum, in contrarias itidem partes factæ, quia motus æqualiter mutantur, ſunt corporibus reciproce proportionales.

Corol. I.

Corpus viribus conjunctis diagonalem parallelogrammi eodem tempore deſcribere, quo latera ſeparatis.



Si corpus dato tempore, vi ſola *M*, ferretur ab *A* ad *B*, & vi ſola *N*, ab *A* ad *C*, compleatur parallelogrammum *ABDC*, & vi utraq; feretur id eodem tempore ab *A* ad *D*. Nam quoniam vis *N* agit ſecundum lineam *AC* ipſi *BD* parallelam, hæc vis nihil mutabit velocitatem accedendi ad lineam illam *BD* a vi altera genitam. Accedet igitur corpus eodem tempore ad lineam *BD* ſive vis *N* imprimatur, ſive non, atq; adeo in fine illius temporis reperietur alicubi in linea illa

Book I: Motion of bodies

Book I, Section I: On the method of first and last ratios

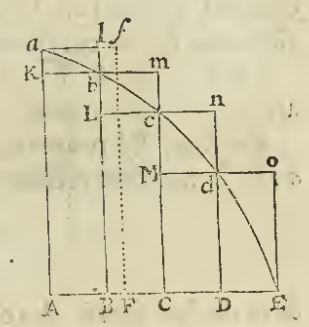
Lemma I: Quantities, and ratios of quantities, which [...] approach nearer to each other than by any given difference, become ultimately equal.

For suppose they are ultimately unequal, and their ultimate difference is D. Then they cannot approach nearer to equality than by that difference.

Book I, Lemma II

Lemma II.

Si in figura quavis AacE *rectis* Aa, AE, *& curva* AcE *comprehenſa, inſcribantur parallelogramma quotcunq;* Ab, Bc, Cd, &c. *ſub baſibus* AB, BC, CD, &c. *æqualibus, & lateribus* Bb, Cc, Dd, &c. *figuræ lateri* Aa *parallelis contenta; & compleantur parallelogramma* aKbl, bLcm, cMdn, &c, *Dein horum parallelogrammorum latitudo minuatur, & numerus augeatur in infinitum: dico quod ultimæ rationes, quas habent ad ſe invicem figura inſcripta* AKbLcMdD, *circumſcripta* Aalbmcndoe, *& curvilinea* AabcdE, *ſunt rationes æqualitatis.*



Nam figuræ inſcriptæ & circumſcriptæ differentia eſt ſumma parallelogrammorum $Kl + Lm + Mn + Do$, hoc eſt (ob æquales omnium baſes) rectangulum ſub unius baſi Kb & altitudinum ſumma Aa, id eſt rectangulum $ABla$. Sed hoc rectangulum, eo quod latitudo ejus AB in infinitum minuitur, fit minus quovis dato. Ergo, per Lemma I, figura inſcripta & circumſcripta & multo magis figura curvilinea intermedia fiunt ultimo æquales. *Q. E. D.*

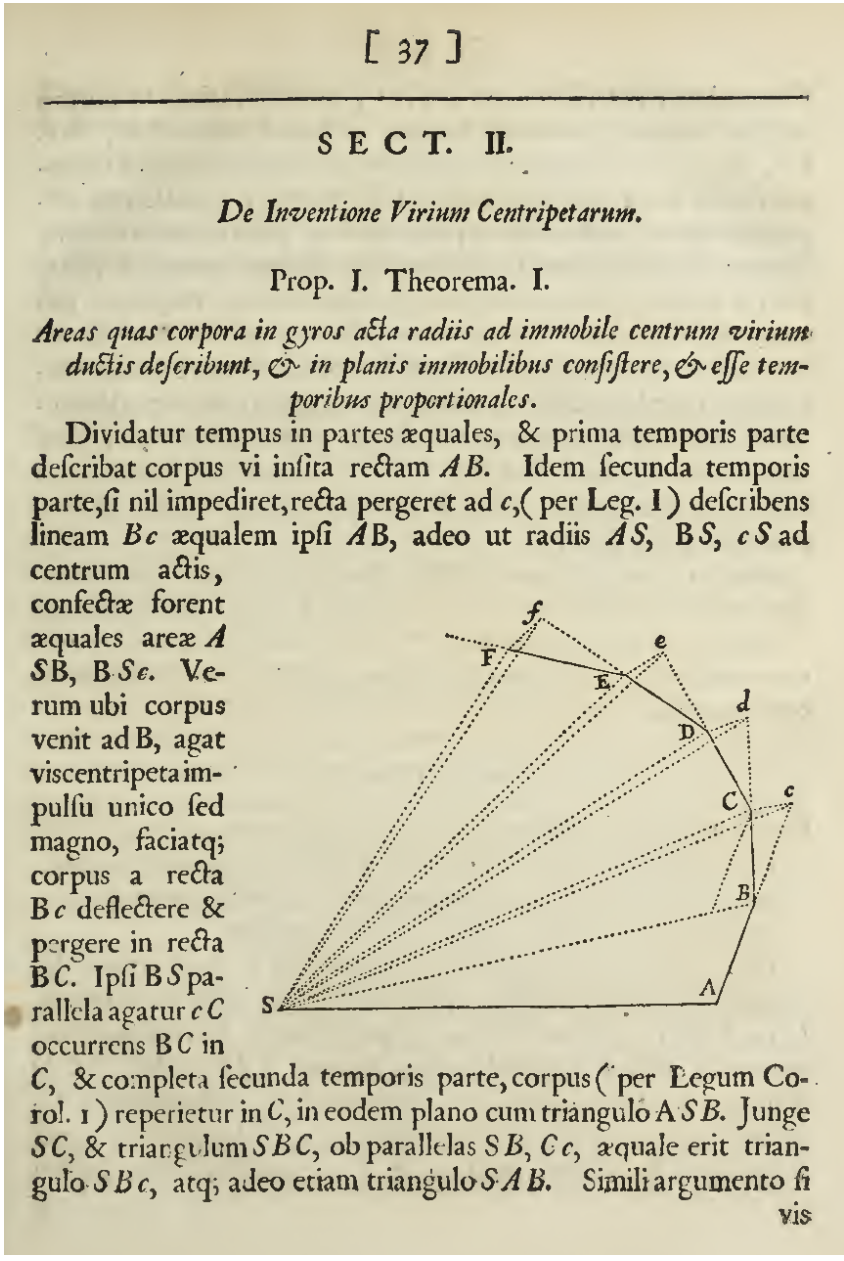
Lemma III.

Eædem rationes ultimæ ſunt etiam æqualitatis, ubi parallelogrammorum latitudines AB, BC, CD, &c. ſunt inæquales, & omnes minuuntur in infinitum.

Sit enim AF æqualis latitudini maximæ, & compleatur parallelogrammum $FAaf$. Hoc erit majus quam differentia figuræ inſcriptæ & ſi quæ circumſcriptæ, at latitudine ſua AF

Lemma II: Ultimate equality of inscribed figure, circumscribed figure, and curved area

Motion under centripetal forces



[37]

SECT. II.

De Inventione Virium Centripetarum.

Prop. I. Theorema. I.

Areas quas corpora in gyros acta radiis ad immobile centrum virium ductis describunt, & in planis immobilibus consistere, & esse temporibus proportionales.

Dividatur tempus in partes æquales, & prima temporis parte describat corpus vi insita rectam *AB*. Idem secunda temporis parte, si nil impediret, recta pergeret ad *c*, (per Leg. I) describens lineam *Bc* æqualem ipsi *AB*, adeo ut radiis *AS*, *BS*, *cS* ad centrum actis, confectæ forent æquales areæ *ASB*, *BSc*. Verum ubi corpus venit ad B, agat viscentripeta impulsu unico sed magno, faciatq; corpus a recta *Bc* deflectere & pergere in recta *BC*. Ipsi *BS* parallela agatur *cC* occurrens *BC* in *C*, & completa secunda temporis parte, corpus (per Legum Corol. 1) reperietur in *C*, in eodem plano cum triangulo *ASB*. Junge *SC*, & triangulum *SBC*, ob parallelas *SB*, *Cc*, æquale erit triangulo *SBc*, atq; adeo etiam triangulo *SAB*. Simili argumento si vis

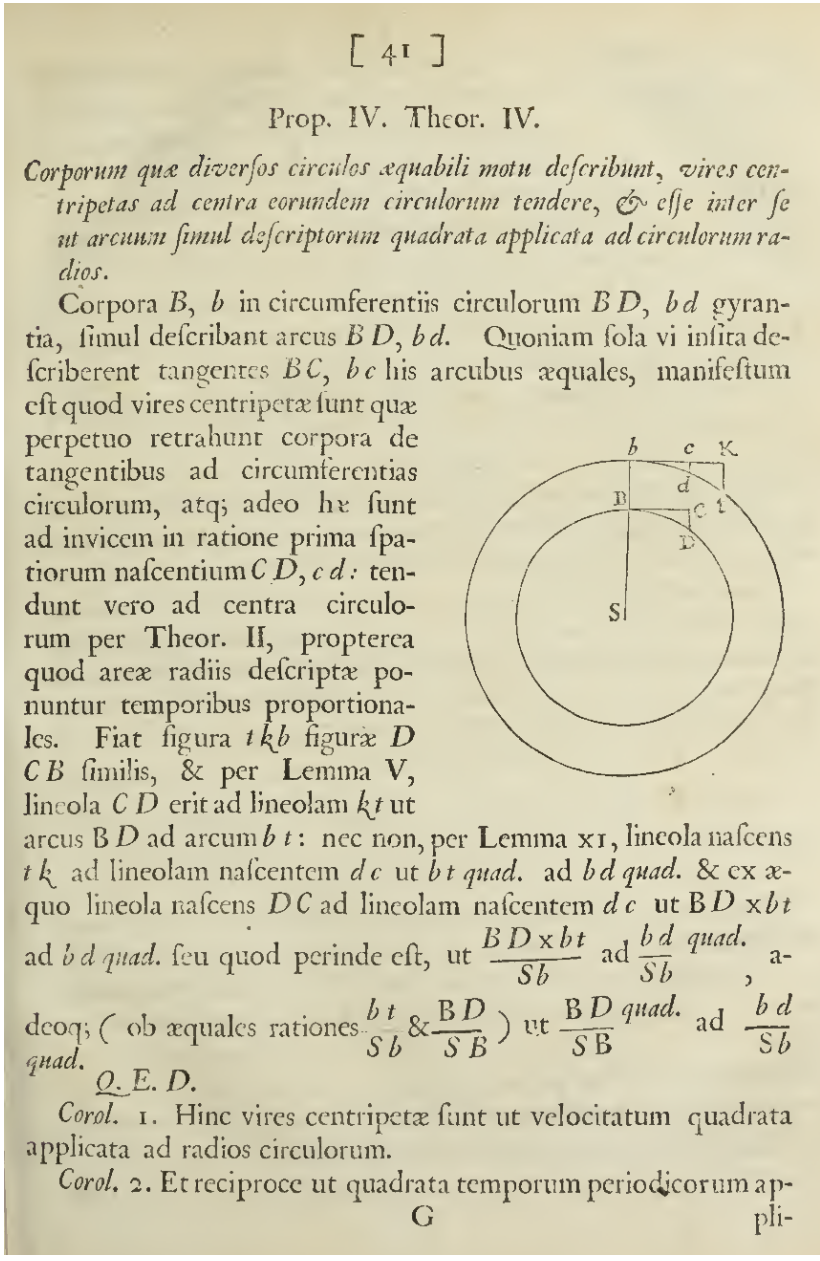
Book I, Section II: Motion under centripetal forces.

Proposition I: Bodies constrained by a central force to orbit a fixed point move in a plane and sweep out equal areas in equal times.

(Kepler's second law)

NB. independent of the 'law of force' involved.

Book I, Section II: Circular motion



[41]

Prop. IV. Theor. IV.

Corporum quæ diverſos circulos æquabili motu deſcribunt, vires centripetas ad centra eorundem circulorum tendere, & eſſe inter ſe ut arcuum ſimul deſcriptorum quadrata applicata ad circulorum radios.

Corpora *B*, *b* in circumferentiis circulorum *B D*, *b d* gyrantia, ſimul deſcribant arcus *B D*, *b d*. Quoniam ſola vi inſita deſcriberent tangentes *B C*, *b c* his arcubus æquales, manifeſtum eſt quod vires centripetæ ſunt quæ perpetuo retrahunt corpora de tangentibus ad circumferentias circulorum, atq; adeo hæ ſunt ad invicem in ratione prima ſpatiorum naſcentium *C D*, *c d*: tendunt vero ad centra circulorum per Theor. II, propterea quod areæ radiis deſcriptæ ponuntur temporibus proportionales. Fiat figura *t k b* figuræ *D C B* ſimilis, & per Lemma V, lineola *C D* erit ad lineolam *k t* ut arcus B *D* ad arcum *b t*: nec non, per Lemma XI, lineola naſcens *t k* ad lineolam naſcentem *d c* ut *b t quad.* ad *b d quad.* & ex æquo lineola naſcens *D C* ad lineolam naſcentem *d c* ut B *D* x *b t* ad *b d quad.* ſeu quod perinde eſt, ut $\frac{B\,D \times b\,t}{S\,b}$ ad $\frac{b\,d\ quad.}{S\,b}$, adeoq; (ob æquales rationes $\frac{b\,t}{S\,b}$ & $\frac{B\,D}{S\,B}$) ut $\frac{B\,D\ quad.}{S\,B}$ ad $\frac{b\,d\ quad.}{S\,b}$. Q. E. D.

Corol. 1. Hinc vires centripetæ ſunt ut velocitatum quadrata applicata ad radios circulorum.

Corol. 2. Et reciproce ut quadrata temporum periodicorum ap-

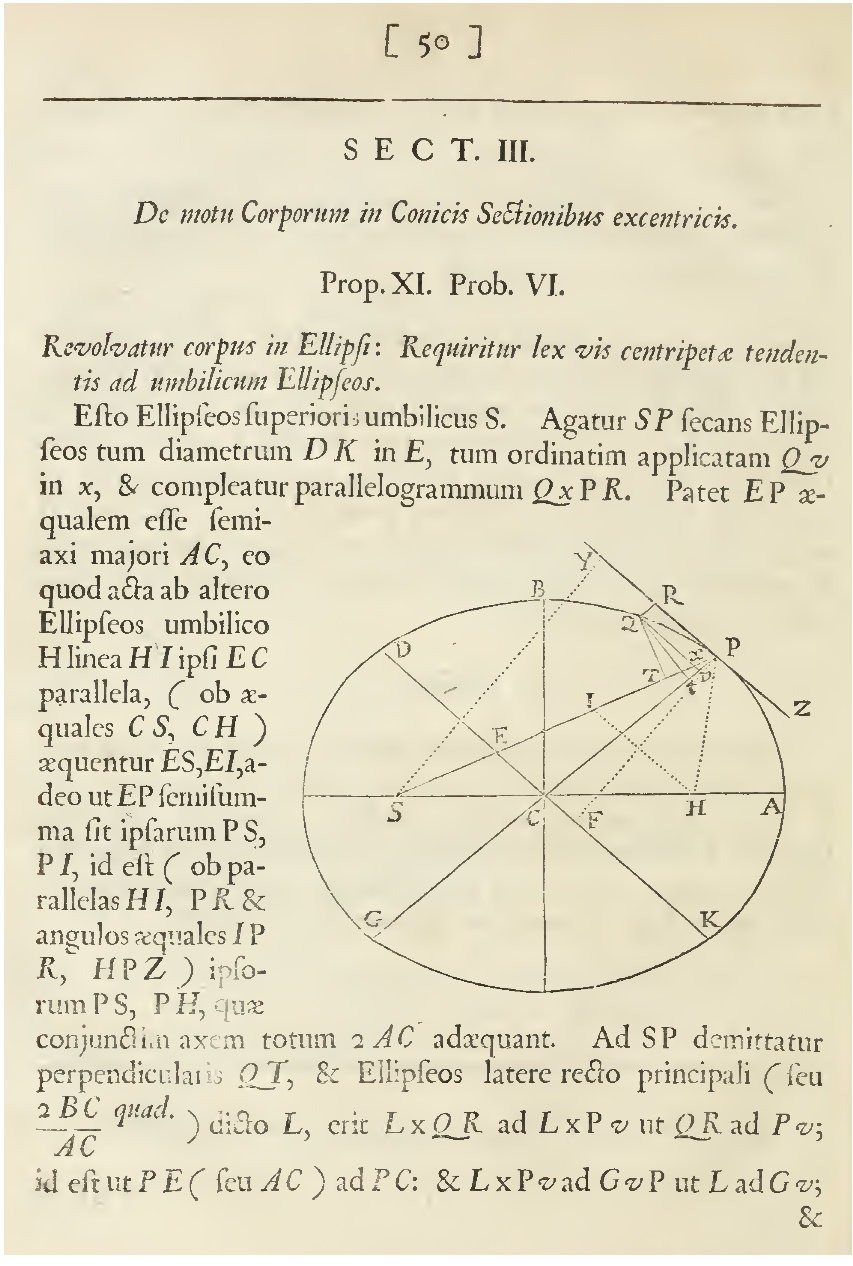
G pli-

Book I, Sect. II, Prop. IV: Motion under centripetal forces: motion in a circle.

Corollary 1: For motion in a circle centripetal force is proportional to $\frac{v^2}{r}$.

Corollary 6: For motion in a circle Kepler's third law implies an inverse square law of force.

Book I, Section III: orbits that are conic sections



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SECT. III.

De motu Corporum in Conicis Sectionibus excentricis.

Prop. XI. Prob. VI.

Revolvatur corpus in Ellipſi: Requiritur lex vis centripetæ tendentis ad umbilicum Ellipſeos.

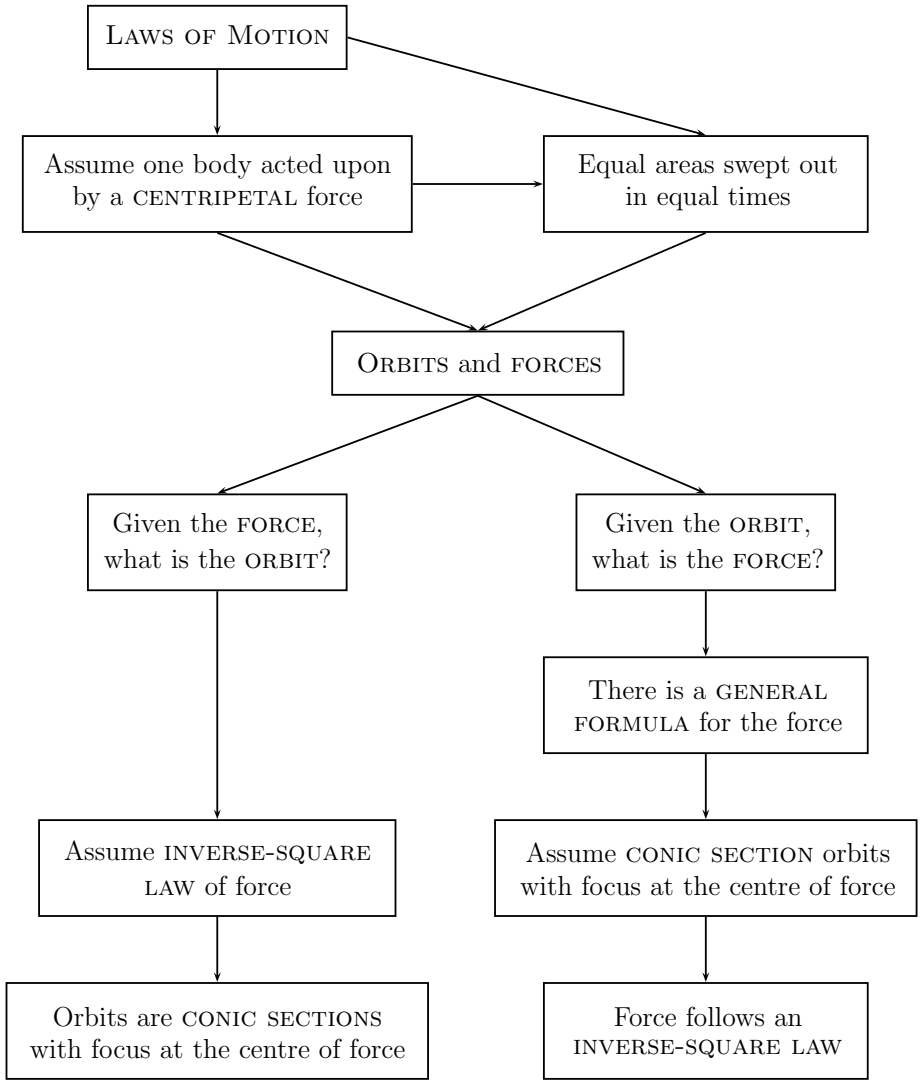
Eſto Ellipſeos ſuperioris umbilicus S. Agatur *SP* ſecans Ellipſeos tum diametrum *DK* in *E*, tum ordinatim applicatam *Qv* in *x*, & compleatur parallelogrammum *QxPR*. Patet *EP* æqualem eſſe ſemiaxi majori *AC*, eo quod acta ab altero Ellipſeos umbilico H linea *HI* ipſi *EC* parallela, (ob æquales *CS*, *CH*) æquentur *ES*, *EI*, adeo ut *EP* ſemiſumma ſit ipſarum *PS*, *PI*, id eſt (ob parallelas *HI*, *PR* & angulos æquales *IPR*, *HPZ*) ipſorum *PS*, *PH*, quæ conjunctim axem totum *2AC* adæquant. Ad *SP* demittatur perpendicularis *QT*, & Ellipſeos latere recto principali (ſeu $\frac{2BC\ quad.}{AC}$) dicto *L*, erit *L* x *QR* ad *L* x *Pv* ut *QR* ad *Pv*; id eſt ut *PE* (ſeu *AC*) ad *PC*: & *L* x *Pv* ad *Gv* *P* ut *L* ad *Gv*; &

Proposition XI: Motion under centripetal forces: Kepler's First Law (orbit is an ellipse with sun at focus) implies an inverse square law of force.

Proposition XII: Motion under centripetal forces: hyperbolic orbit implies an inverse square law of force.

Proposition XIII: Motion under centripetal forces: parabolic orbit implies an inverse square law of force.

Book I, Sections II and III summarised


Laws of Motion
Assume one body acted upon by a centripetal force
Equal areas swept out in equal times
Orbits and forces
Given the force, what is the orbit?
Given the orbit, what is the force?
There is a general formula for the force
Assume inverse-square law of force
Assume conic section orbits with focus at the centre of force
Orbits are conic sections with focus at the centre of force
Force follows an inverse-square law

(Adapted from Colin Pask, *Magnificent Principia*, Prometheus Books, 2013, p. 218)

Book I, later sections

More mechanics of motion:

- converses: an inverse square law of force implies that orbits are conic sections;
- trajectories;
- much more besides.

All treated geometrically

Books II and III

Book II: Motion of bodies in resisting media:

Conclusion: "... it is manifest that the planets are not carried round in corporeal vortices ..." (Scholium to Proposition LIII)

Book III: The system of the world:

- Reconciliation of observation and theory
- Shape of the earth (correct?)
- Motion of the moon (wrong)
- Prediction of tides
- Comets

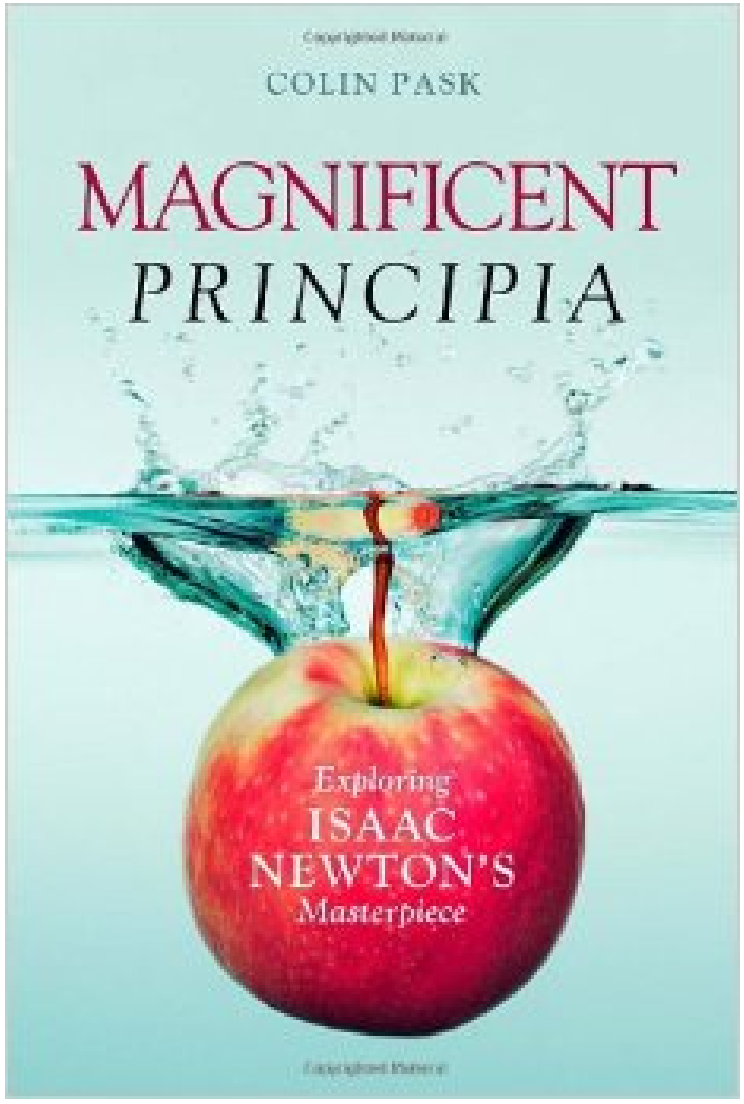
Influence of the *Principia*

Principia showed how mathematical methods could be used to study physical, especially but not exclusively, cosmological phenomena.

New ways of thinking: for example the method of ultimate ratios, though expressed geometrically, came close to a modern concept of limits.

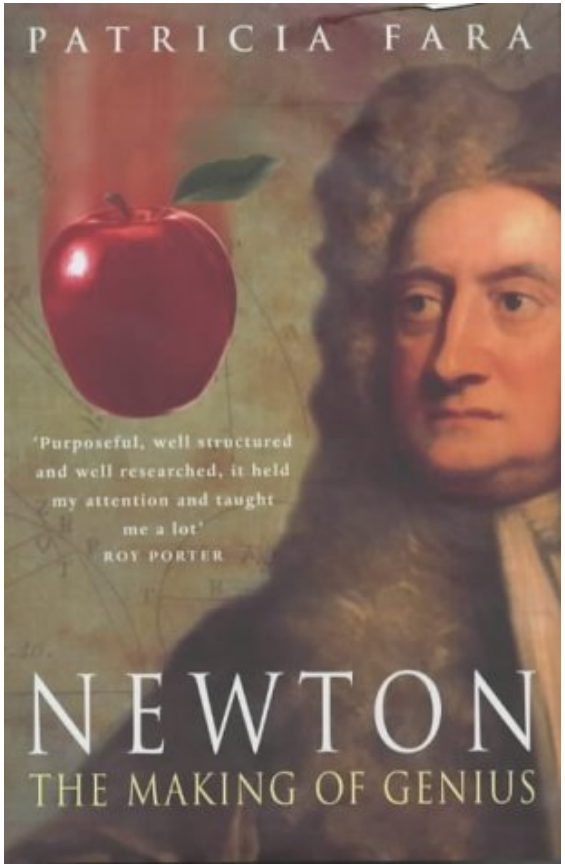
Predictions could be verified by observation and experiment — verified (after some controversy) in the case of the shape of the earth, contradicted in the case of the motion of the moon.

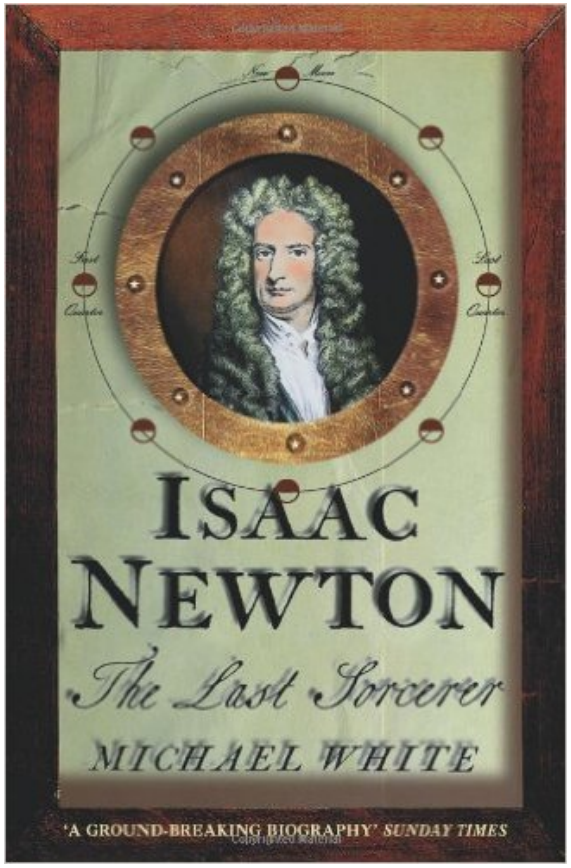
For more on the *Principia*...

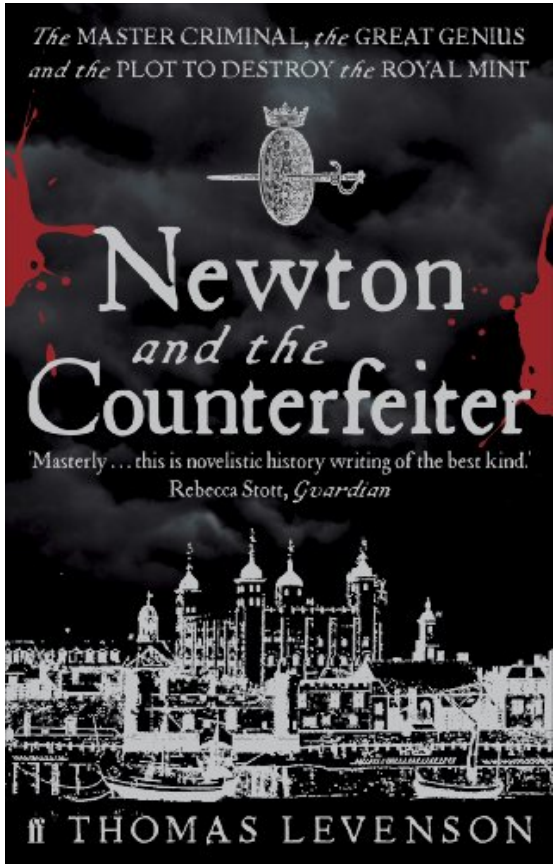


(Colin Pask, *Magnificent Principia*, Prometheus Books, 2013)

Three (very different) books among many...


PATRICIA FARA
'Purposeful, well structured and well researched, it held my attention and taught me a lot'
ROY PORTER
NEWTON
THE MAKING OF GENIUS




The MASTER CRIMINAL, the GREAT GENIUS and the PLOT TO DESTROY the ROYAL MINT
Newton
and the
Counterfeiter
'Masterly . . . this is novelistic history writing of the best kind.'
Rebecca Stott, Guardian
THOMAS LEVENSON

And a lecture given at Gresham College:
https://www.gresham.ac.uk/lectures-and-events/isaac-newtons-world

BO1 History of Mathematics
Lecture VI
Successes of and difficulties with the calculus: the 18th-century beginnings of 'rigour'

MT 2022 Week 3

Summary

- Publication and acceptance of the calculus
- Some successes of the calculus
- Functions
- Problems with the calculus
- Some responses: beginnings of ‘rigour’ in Analysis

Reminder: a comparison from lecture IV

Newton (1664–65):	Leibniz (1673–76):
rules for quadrature	rules for quadrature
rules for tangents	rules for tangents
'fundamental theorem'	'fundamental theorem'
dot notation	'modern' notation
physical intuition:	algebraic intuition
rates of change	rules and procedures
PROBLEM:	PROBLEM:
vanishing quantities o	vanishing quantities du, dv, ...

Newton's publication (or not) of his calculus

1669: 'De analysi' shown to Barrow and Collins

1671: 'Treatise on fluxions and infinite series' withdrawn before publication

1676: two long letters to Leibniz, plus a coded message

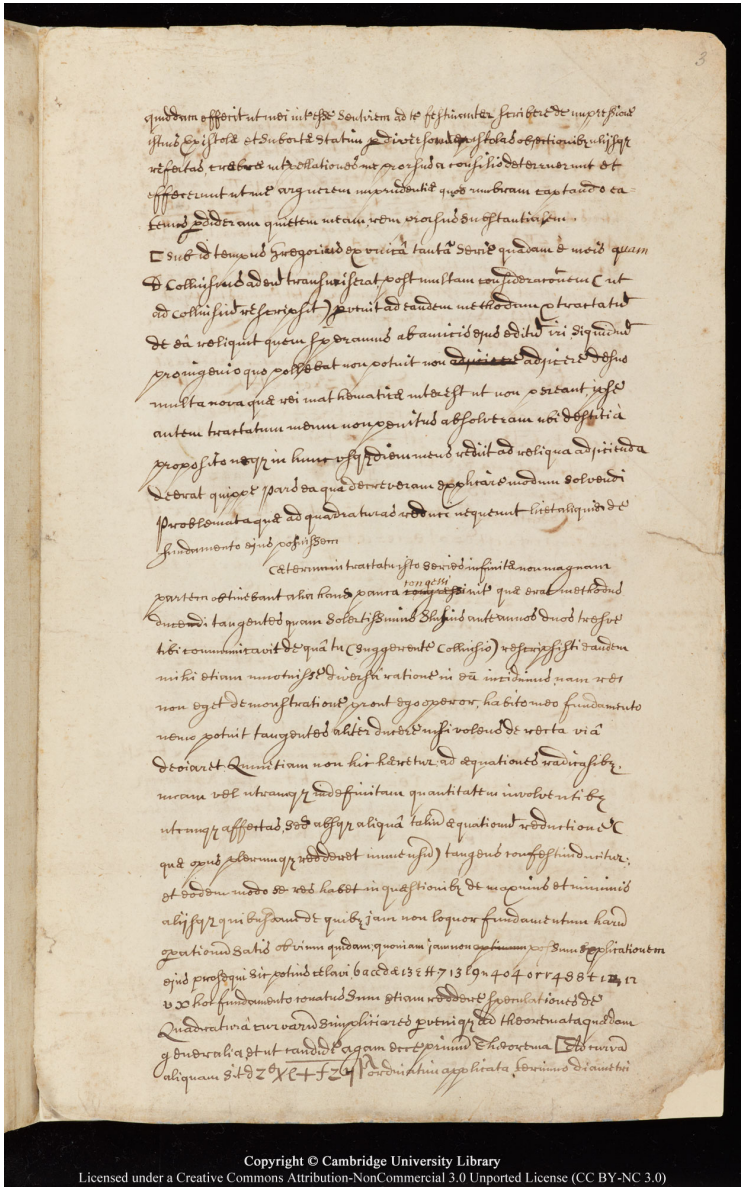
1685: partial publication of the letters to Leibniz by Wallis in his *Treatise of algebra*

1693: further partial publication by Wallis in his *Opera mathematica*

1694: some material written up by David Gregory, but not published

1704: 'Treatise of quadrature' appended to published *Opticks*

Newton's coded message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CUL MS Add.3977 f. 3r

Letter from Isaac Newton to Henry Oldenburg, 24 October 1676 ('Epistola posterior')

"The foundation of these operations is evident enough, in fact; but because I cannot proceed with the explanation of it now, I have preferred to conceal it thus: 6*accdae*13*eff*7*i*3*l*9*n*4*o*4*qrr*4*s*8*t*12*vx*."

"Data aequatione quotcunque fluentes quantitates involvente, fluxiones invenire: et vice versa."

= "Given an equation involving any number of fluent quantities, to find the fluxions: and vice versa."

Leibniz's publication of his calculus

1680s: Papers in *Acta eruditorum* (journal founded 1682)

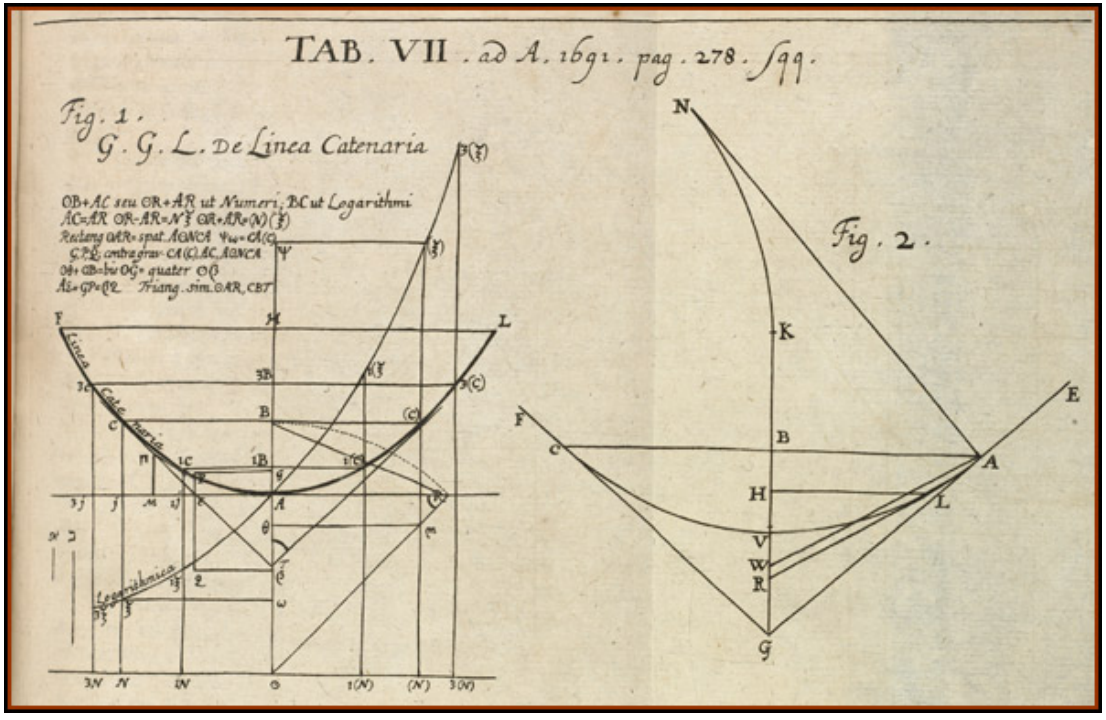
1691: Bernoulli brothers (Johann and Jacob) begin to apply Leibniz' methods

1696: Exposition by L'Hôpital based on teachings of Johann Bernoulli

Challenge problems

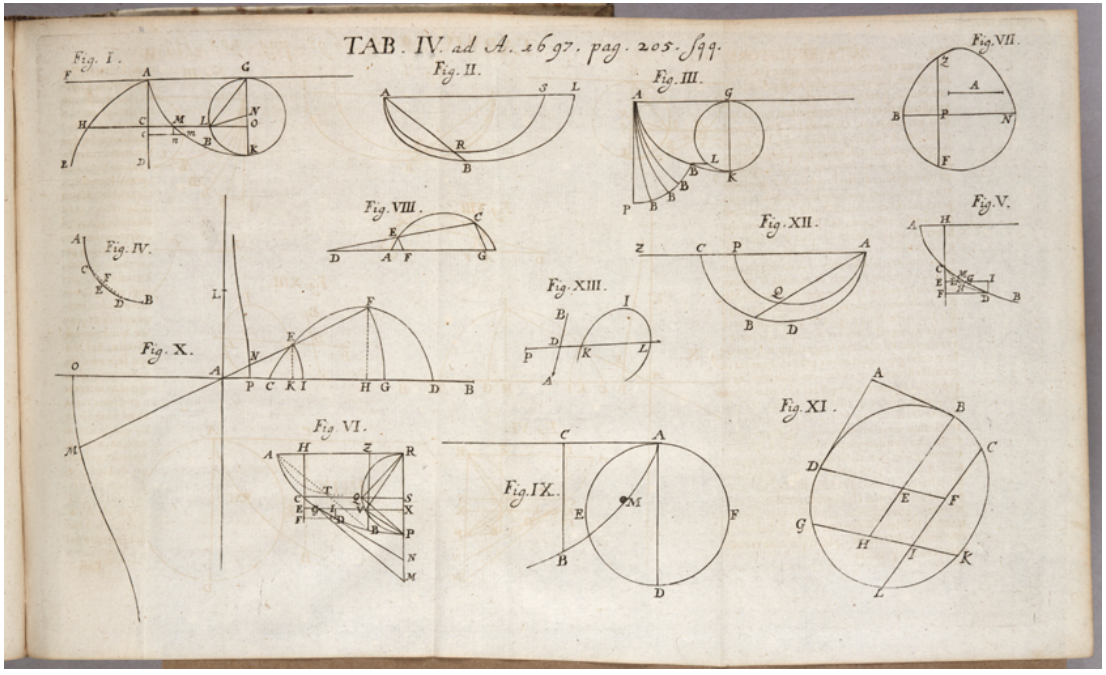
1687: Isochrone — curve of uniform descent
(posed by Leibniz; solved by Jacob Bernoulli)

1691: Catenary — curve of a hanging chain
(posed by Jacob Bernoulli; solved by Johann Bernoulli, Huygens, Leibniz)



Leibniz' and Huygens' solutions, *Acta eruditorum*, 1691.

Challenge problems



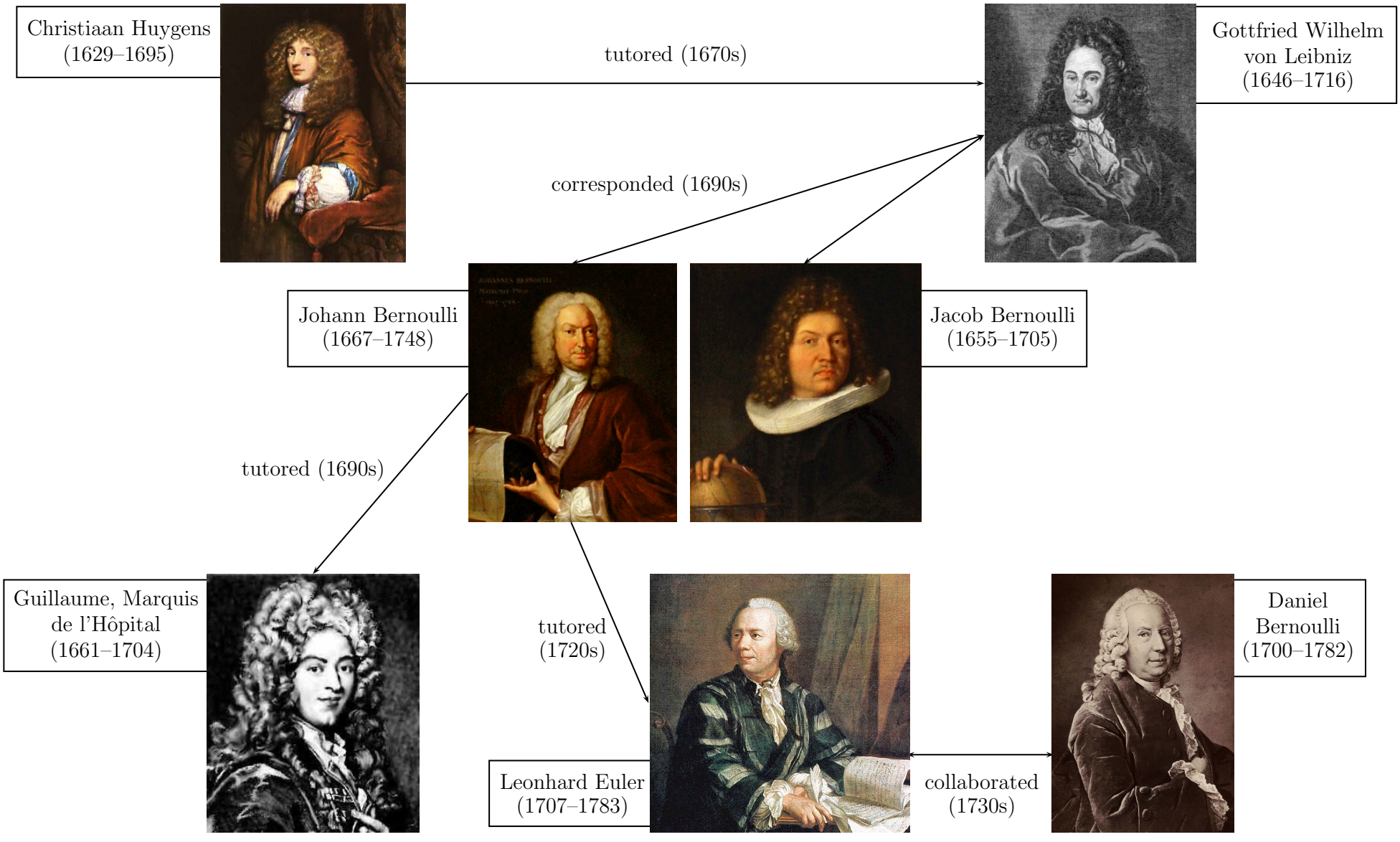
Solutions by Johann & Jacob Bernoulli, l'Hospital, and Newton, *Acta eruditorum*, 1696.

1696: Brachistochrone — curve of fastest descent (posed by Johann Bernoulli; shown to be cycloid by Jacob Bernoulli, Leibniz, Newton, l'Hôpital)

1697: Isoperimeter problems — find curve of given length that maximises a certain integral (classical problem; variant posed by Jacob Bernoulli, solved by him 1701)

And many others

People and connections



Leonhard Euler (1707–1783): a major 18th-century figure

1707: Euler born in Basel

1720: attended University of Basel, taught by Johann Bernoulli

1727: left Basel for Saint Petersburg with Daniel Bernoulli

1741: invited to Frederick the Great's new Academy in Berlin

1766: returned to St Petersburg

1783: died in St Petersburg

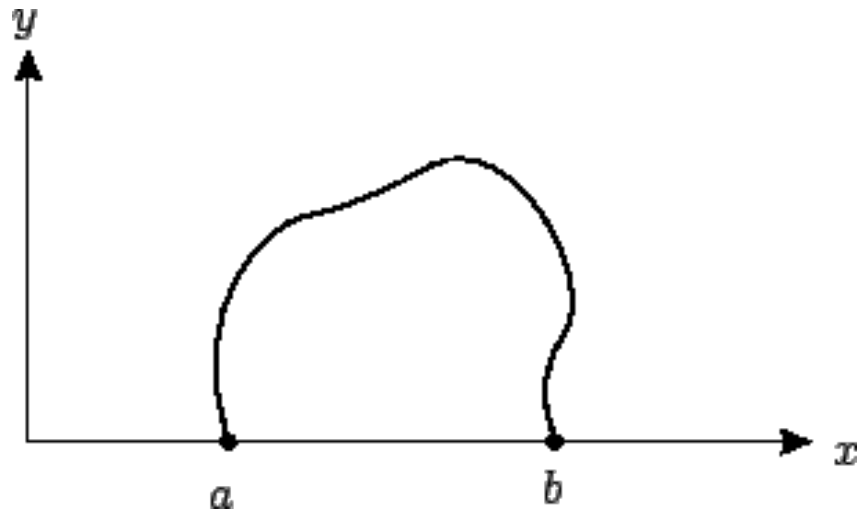
Influence of the challenge problems

These challenge problems and others helped to

- consolidate and validate Leibnizian calculus
- introduce new questions about ‘functions’, ‘differentiability’, ‘continuity’, ...

Functions: isoperimeter problem

Classical Problem (Virgil's *Aeneid*): Find the closed curve of given length L that maximises the area enclosed.

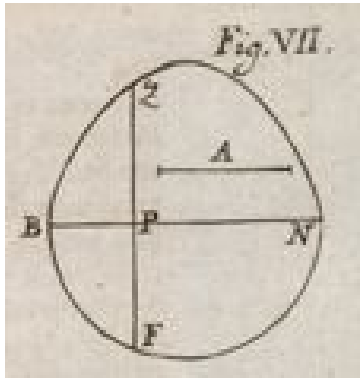


Modern Formulation: Find a function f and corresponding curve $y = f(x)$ between $(a, 0)$ and $(b, 0)$ of given length L (where $L > b - a$) that maximises the area beneath it.

But what is meant by 'function'?

Functions: isoperimeter problem

Isoperimeter problem posed by Jacob Bernoulli to Johann Bernoulli, May 1697, verbally and geometrically (ratio and proportion)



December 1697: problem rephrased by Johann in terms of powers

Solved by Johann in June 1698; published in 1706, with problem phrased in terms of functions (undefined)

In 1718, gave the following definition:

> *Here one calls a function of a variable magnitude, a quantity composed in any manner possible from this variable magnitude and constants.*

(See *Mathematics emerging*, §9.1.1.)

Functions: the wave equation

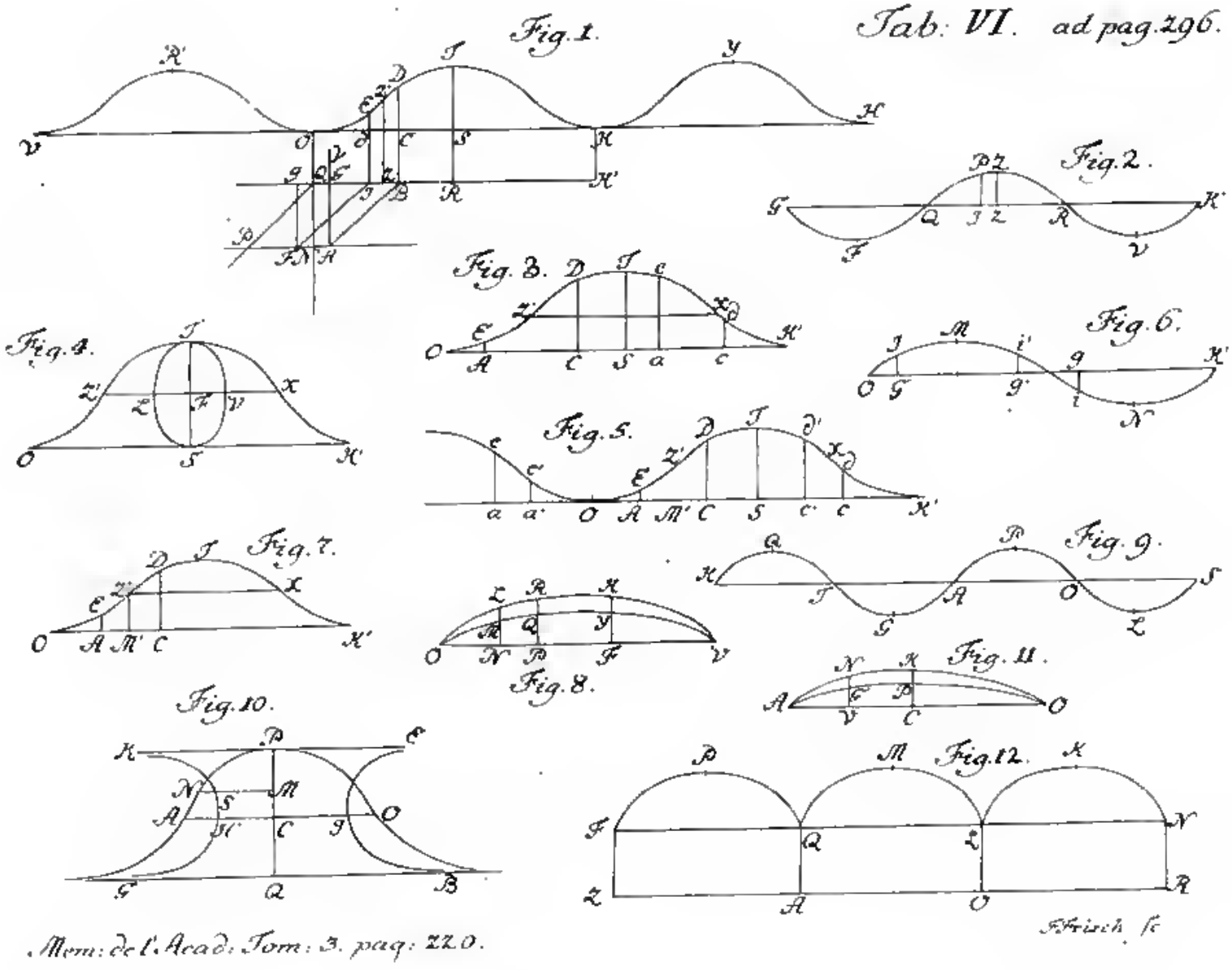
Another success of calculus: the wave equation

$$\frac{\partial^2 y}{\partial s^2} = c^2 \frac{\partial^2 y}{\partial t^2}$$

Solved by d'Alembert (1747) and Euler (1748) with solutions of the form

$$y = \Psi(s + ct) - \Phi(s - ct).$$

Functions: the wave equation



Functions: the wave equation

But which ‘functions’ are admissible as solutions?

Must they be

- continuous?
- differentiable?
- ... whatever these mean ...

What is a function?

Euler's definition of a function (1748):

> *A function of a variable quantity is an analytic expression composed in any way from that variable quantity and from numbers or constant quantities.*
>
> ...
>
> *Functions are divided into algebraic and transcendental; the former are those composed by algebraic operations alone, but the latter are those in which transcendental operations are involved.*

L. Euler: *Introductio in analysin infinitorum* (1748) [*Introduction to the analysis of the infinite*], available in translation, Springer-Verlag, 1988.

What is a function?

Euler's new definition of a function (1755):

> *Moreover, the quantities that depend in this way on others, so that the latter having changed, they themselves also undergo change, are usually called functions; which name opens up most generally all the ways in which one quantity may be determined from others involved with it.*

L. Euler: *Institutiones calculi differentialis* [*Foundations of differential calculus*] (1755)

What is a function?

In fact, this question took a long time to settle.

Nineteenth-century authors were split between those who preferred Euler's definition of 1748 and that of 1755 (see *Mathematics emerging*, §9.3).

The idea of a function as a mapping began to emerge at the end of the nineteenth century, in, for example, Dedekind's *Was sind und was sollen die Zahlen?* (1888), a text that we will come back to in a later lecture.

[For a list of different definitions of functions, ranging from 1718 to 1939, see: Dieter Rüthing, Some definitions of the concept of function from Joh. Bernoulli to N. Bourbaki, *The Mathematical Intelligencer* **6**(4) (1984) 72–77]

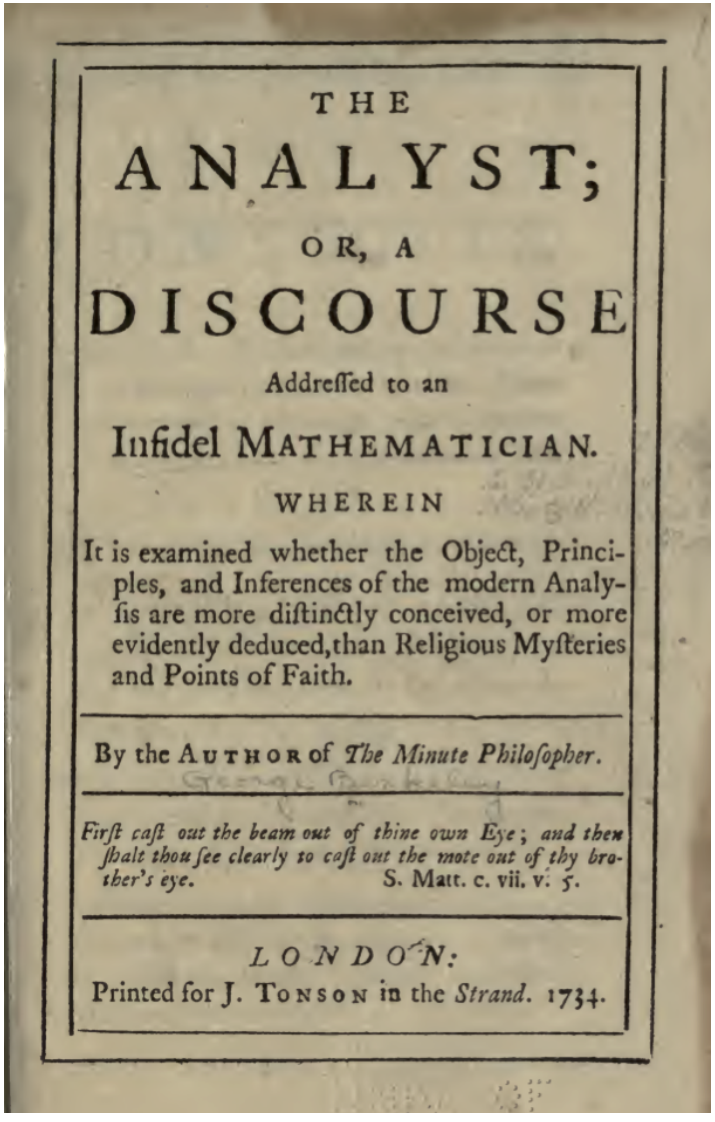
More problems: infinitely small quantities

Thomas Hobbes, *Six lessons to the Professors of Mathematicks* (1656):

> *The least Altitude is Somewhat or Nothing. If Somewhat, then the first character of your Arithmeticall Progression must not be zero;*
>
> . . .
>
> *If Nothing, then your whole figure is without Altitude and consequently your Understanding nought.*

Wallis tried to provide further explanation in his *Due correction for Mr. Hobbes* (1656), but wasn't too concerned by the problems

More problems: infinitely small quantities



THE

ANALYST;

OR, A

DISCOURSE

Addreſſed to an

Infidel MATHEMATICIAN.

WHEREIN

It is examined whether the Object, Principles, and Inferences of the modern Analyſis are more diſtinctly conceived, or more evidently deduced, than Religious Myſteries and Points of Faith.

By the AUTHOR of *The Minute Philoſopher.*

Firſt caſt out the beam out of thine own Eye; and then ſhalt thou ſee clearly to caſt out the mote out of thy brother's eye. S. Matt. c. vii. v. 5.

LONDON:

Printed for J. TONSON in the *Strand.* 1734.

George Berkeley (1734)

Qu. 43: Whether an algebraist, fluxionist, geometrician, or demonstrator of any kind can expect indulgence for obscure principles or incorrect reasoning? And whether an algebraical note or species can at the end of a process be interpreted in a sense which could not have been substituted for it at the beginning?

Qu. 45: Whether, although geometry be a science, and algebra allowed to be a science, and the analytical a most excellent method, in the application nevertheless of the analysis to geometry, men may not have admitted false principles and wrong methods of reasoning?

Some responses to the difficulties

Guillaume Marquis de l'Hôpital, *Analyse des infiniment petits* (1696)

Colin Maclaurin, *A treatise of fluxions* (1742)

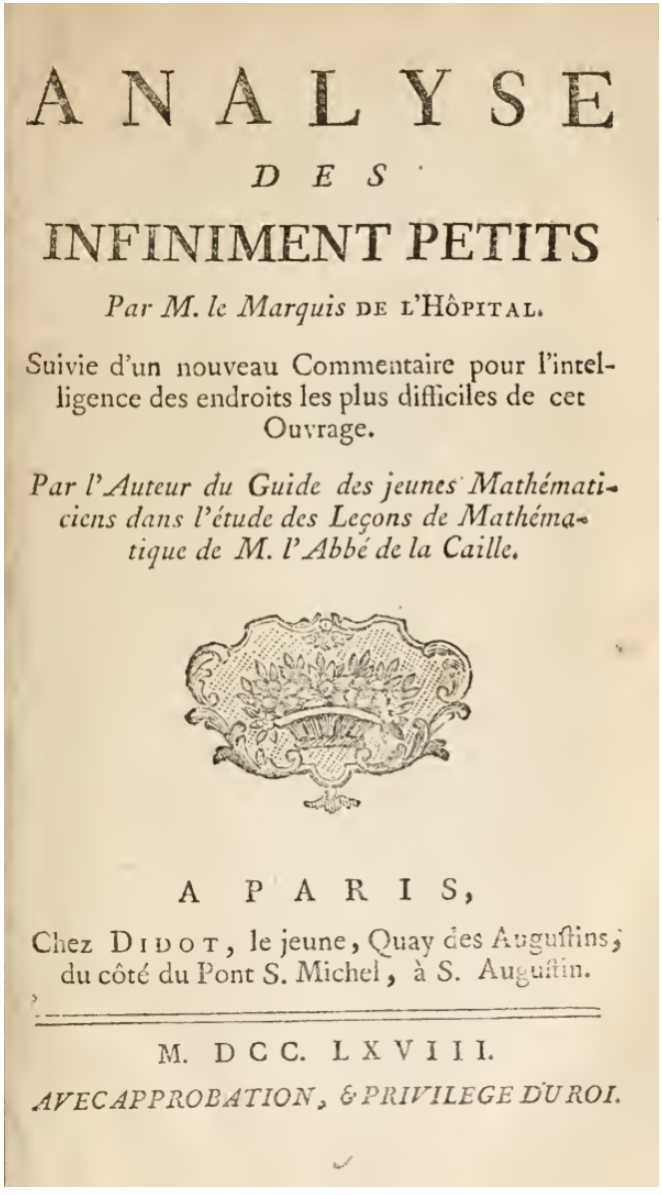
Maria Agnesi, *Instituzioni analitiche ad uso della gioventù italiana* (1748)

Leonhard Euler, *Institutiones calculi differentialis* (1755)

John Landen, *A discourse concerning the residual analysis* (1758)

Joseph-Louis Lagrange, *Théorie des fonctions analytiques* (1797)

Responses to the difficulties: l'Hôpital



ANALYSE
DES
INFINIMENT PETITS

Par M. le Marquis DE L'HÔPITAL.

Suivie d'un nouveau Commentaire pour l'intelligence des endroits les plus difficiles de cet Ouvrage.

Par l'Auteur du Guide des jeunes Mathématiciens dans l'étude des Leçons de Mathématique de M. l'Abbé de la Caille.

A PARIS,

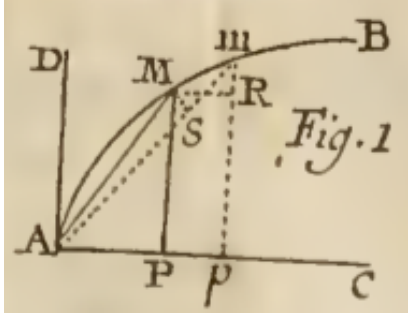
Chez DIDOT, le jeune, Quay des Augustins, du côté du Pont S. Michel, à S. Augustin.

M. DCC. LXVIII.

AVEC APPROBATION, & PRIVILEGE DU ROI.

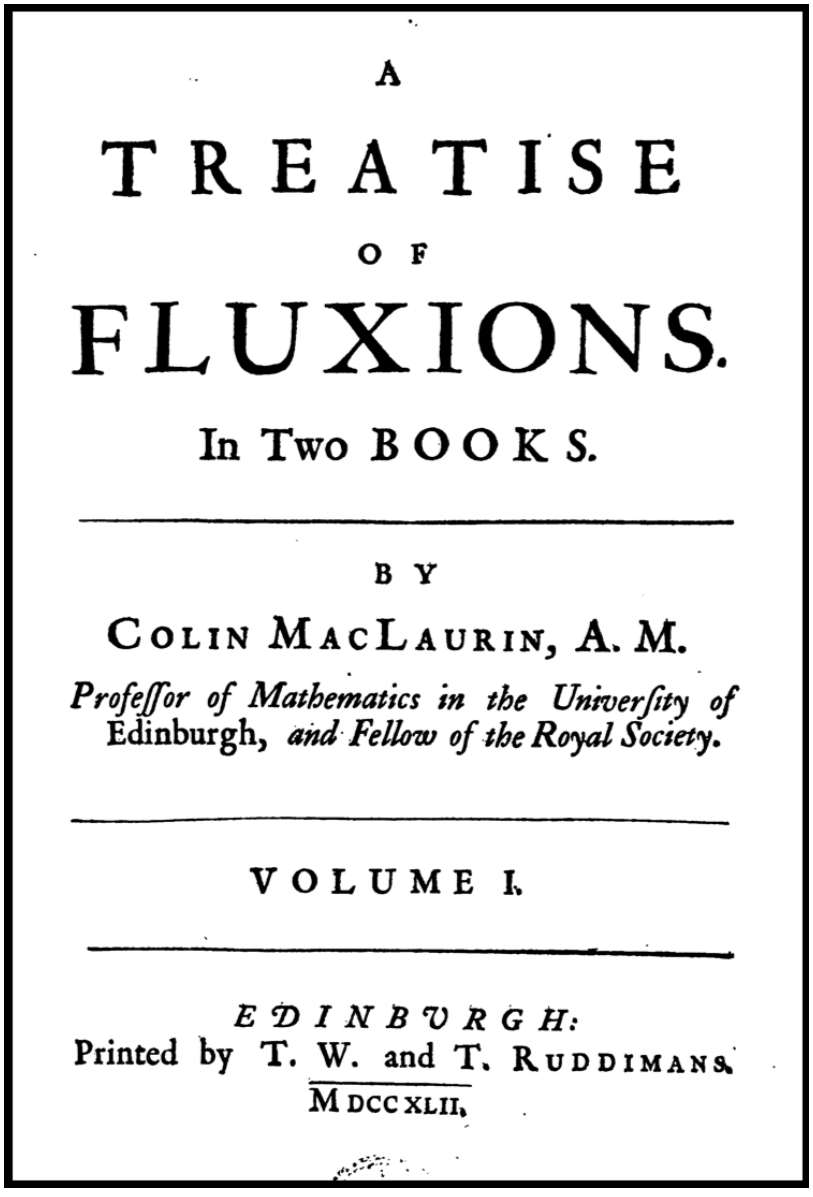
Guillaume, Marquis de l'Hôpital (1696)

Definition. The infinitely small part whereby a variable quantity is continually increased or decreased, is called the differential of that quantity.



Postulate. Grant that two quantities whose difference is an infinitely small quantity may be taken (or used) indifferently for each other: or (which is the same thing) that a quantity which is increased or decreased only by an infinitely small quantity may be considered as remaining the same.

Responses to the difficulties: Maclaurin



A

TREATISE

OF

FLUXIONS.

In Two BOOKS.

BY

COLIN MACLAURIN, A. M.

Professor of Mathematics in the University of Edinburgh, *and Fellow of the Royal Society.*

VOLUME I.

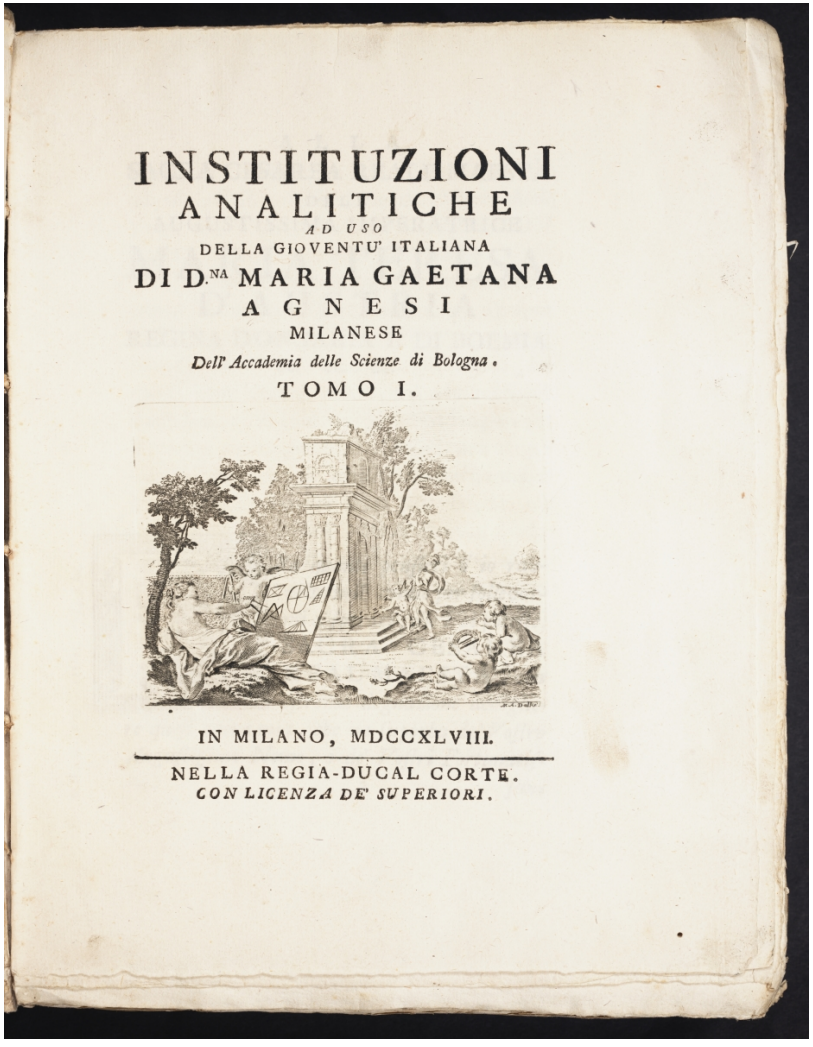
EDINBURGH:
Printed by T. W. and T. RUDDIMANS.
MDCCXLII.

Colin Maclaurin (1742)

- written in direct response to Berkeley
- attempted to prove all propositions of calculus by classical Archimedean methods

  ('double contradiction': derive a contradiction from the assumption that $a > b$; derive a contradiction from the assumption that $b > a$; then it must be the case that $a = b$).

Responses to the difficulties: Agnesi

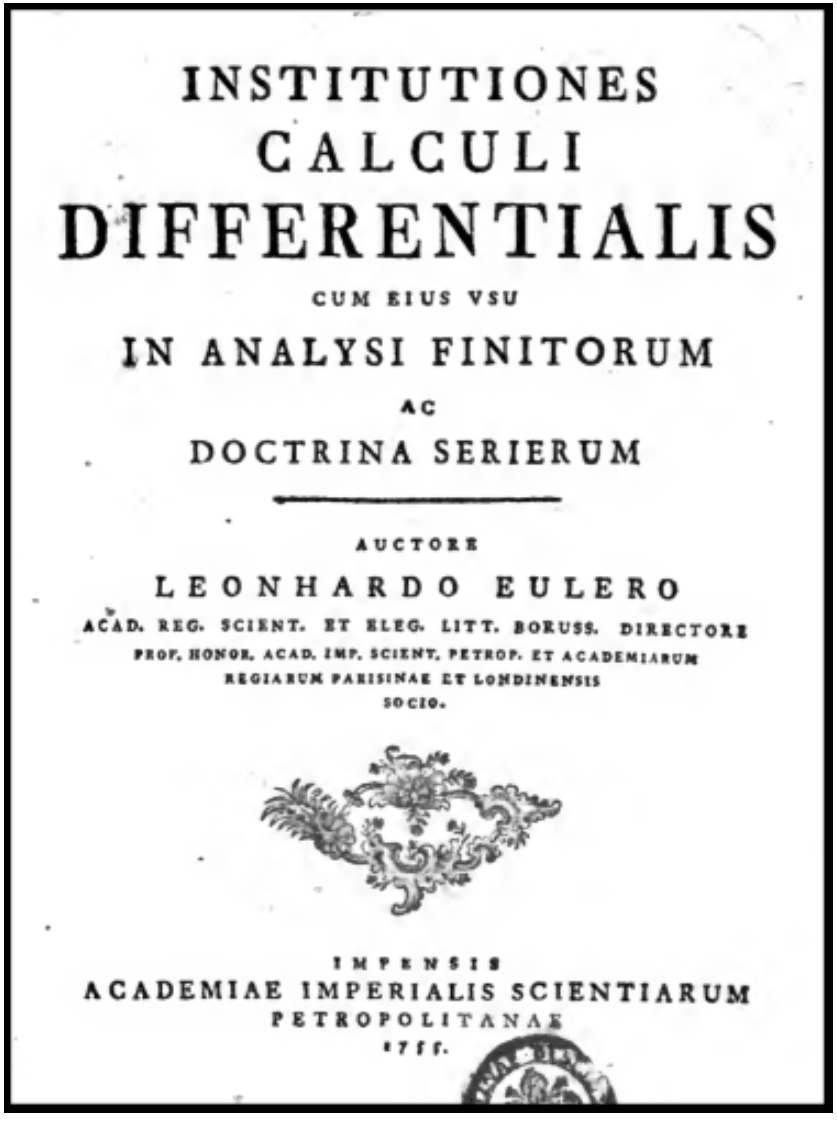

INSTITUZIONI
ANALITICHE
AD USO
DELLA GIOVENTU' ITALIANA
DI D.NA MARIA GAETANA
AGNESI
MILANESE
Dell' Accademia delle Scienze di Bologna.
TOMO I.

IN MILANO, MDCCXLVIII.
NELLA REGIA-DUCAL CORTE.
CON LICENZA DE' SUPERIORI.

Maria Agnesi (1748)

- systematic introduction to calculus
- sought to place calculus on an algebraic basis
- translated into English (1801)

Responses to the difficulties: Euler



INSTITUTIONES
CALCULI
DIFFERENTIALIS
CUM EIUS VSU
IN ANALYSI FINITORUM
AC
DOCTRINA SERIERUM

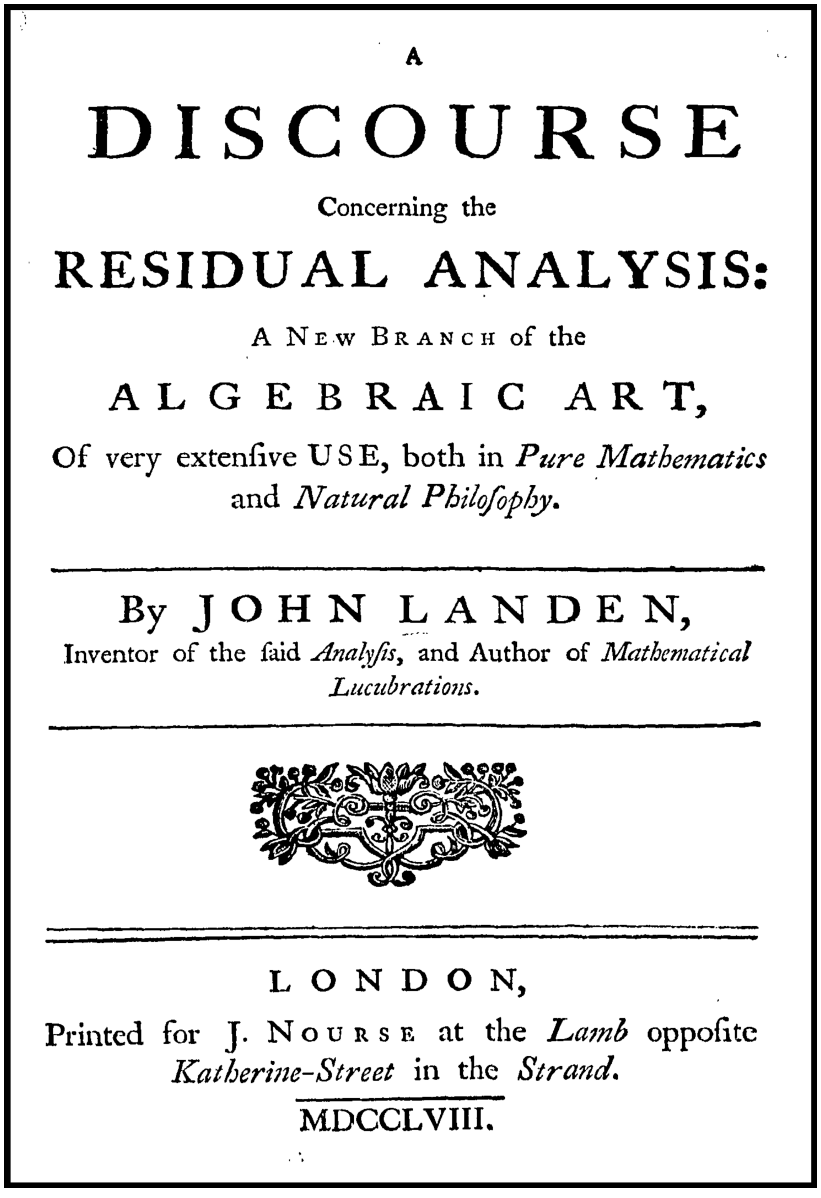
AUCTORE
LEONHARDO EULERO
ACAD. REG. SCIENT. ET ELEG. LITT. BORUSS. DIRECTORE
PROF. HONOR. ACAD. IMP. SCIENT. PETROP. ET ACADEMIARUM
REGIARUM PARISINAE ET LONDINENSIS
SOCIO.

IMPENSIS
ACADEMIAE IMPERIALIS SCIENTIARUM
PETROPOLITANAE
1755.

Leonard Euler (1755):

> *An infinitely small quantity is nothing other than a vanishing quantity, and is therefore really equal to 0.*
>
> . . .
>
> *If there occur different infinitely small quantities dx and dy, although both are equal to 0, nevertheless their ratio is not constant.*

Responses to the difficulties: Landen



A

DISCOURSE

Concerning the

RESIDUAL ANALYSIS:

A NEW BRANCH of the

ALGEBRAIC ART,

Of very extenſive USE, both in *Pure Mathematics* and *Natural Philoſophy*.

By JOHN LANDEN,

Inventor of the ſaid *Analyſis*, and Author of *Mathematical Lucubrations*.

LONDON,

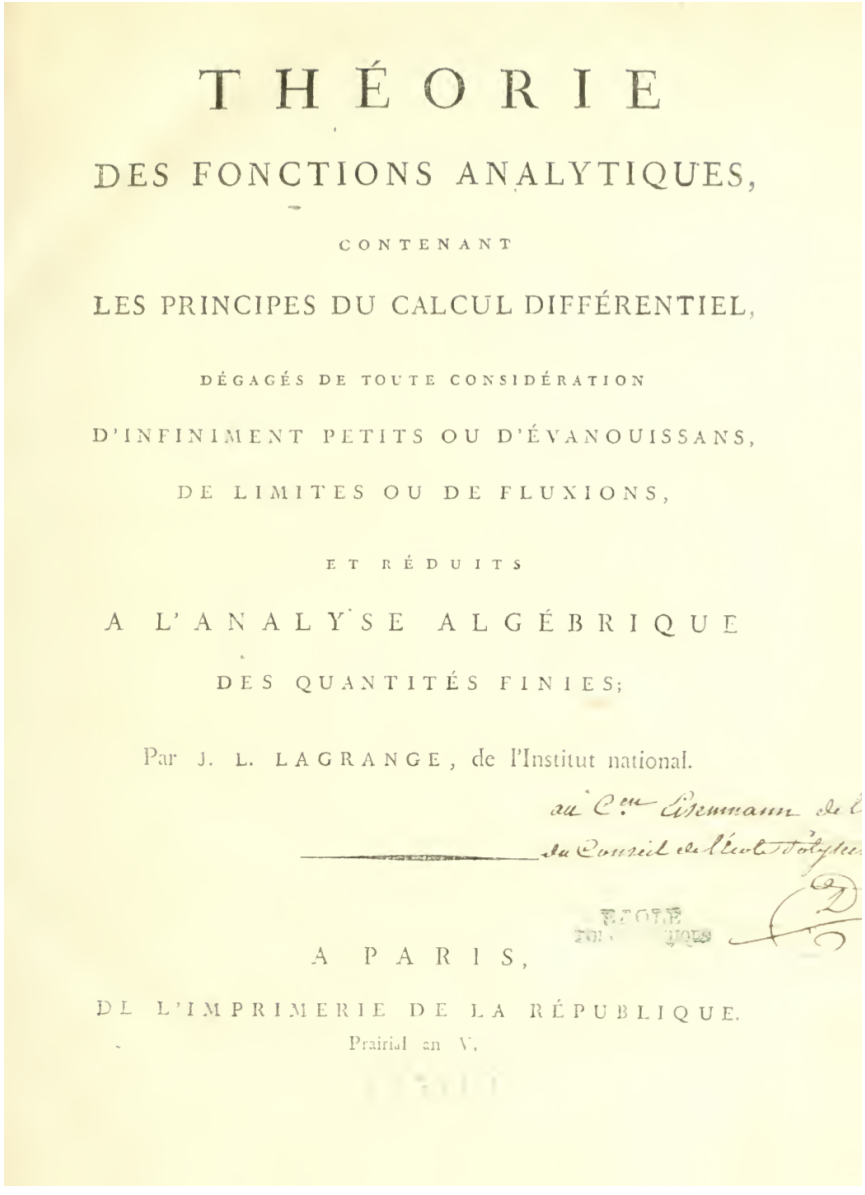
Printed for J. NOURSE at the *Lamb* oppoſite *Katherine-Street* in the *Strand*.

MDCCLVIII.

John Landen (1758)

- 'Fluxions are not immediately applicable to algebraic quantities ...'
- attempted a purely algebraic development of calculus

Responses to the difficulties: Lagrange



THÉORIE

DES FONCTIONS ANALYTIQUES,

CONTENANT

LES PRINCIPES DU CALCUL DIFFÉRENTIEL,

DÉGAGÉS DE TOUTE CONSIDÉRATION

D'INFINIMENT PETITS OU D'ÉVANOUISSANS,

DE LIMITES OU DE FLUXIONS,

ET RÉDUITS

A L'ANALYSE ALGÉBRIQUE

DES QUANTITÉS FINIES;

Par J. L. LAGRANGE, de l'Institut national.

A PARIS,

DE L'IMPRIMERIE DE LA RÉPUBLIQUE.

Prairial an V.

Joseph-Louis Lagrange (1797)

Another attempt to avoid 'infinitely small quantities'

(by taking functions to be defined by power-series expansions)

‘Rigour’ and ‘professionalisation’

Finally, note the increasing ‘professionalisation’ of mathematics in the eighteenth century:

- more university positions
  - Jacob Bernoulli in Basel;
  - Johann Bernoulli in Groningen, then Basel;
- new Academies in St Petersburg and Berlin provided positions with salaries
  - Euler at St Petersburg and Berlin;
  - d’Alembert in Paris;
  - Lagrange followed Euler to Berlin, later went to Paris;
- each Academy had its own ‘Mémoires’ or ‘Transactions’ enabling wider (and sometimes faster) circulation of new ideas.

BO1 History of Mathematics
Lecture VII
Infinite series

MT 2022 Week 4

Summary

- A non-Western prelude
- Newton and the Binomial Theorem
- Other 17th century discoveries
- Ideas of convergence
- Much 18th century progress: power series
- Doubts — and more on convergence

The Kerala School

Flourished in Southern India from the 14th to the 16th centuries, working on mathematical and astronomical problems

Names associated with the school: Narayana Pandita, Madhava of Sangamagrama, Vatasseri Parameshvara Nambudiri, Kelallur Nilakantha Somayaji, Jyeṣṭhadeva, Achyuta Pisharati, Melpathur Narayana Bhattathiri, Achyutha Pisharodi, Narayana Bhattathiri

Treatises on arithmetic, algebra, geometry, inc. methods for approximation of roots of equations, discussion of magic squares, infinite series, ...

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Treatises on arithmetic, algebra, geometry, inc. methods for approximation of roots of equations, discussion of magic squares, infinite series, ...

Tantrasamgraha (1501)

Completed by Kelallur Nilakantha Somayaji (1444–1544) in 1501; concerns astronomical computations

Keralan series

Infinite series for trigonometric functions appear in Sanskrit verse in an anonymous commentary on the *Tantrasamgraha*, entitled the *Tantrasamgraha-vyakhya*, of c. 1530:

इष्टज्यात्रिज्ययोर्घाताट् कोटघाप्तं प्रथमं फलम् ।
ज्यावर्गं गुणकं कृत्वा कोटिवर्गं च हारकम् ॥
प्रथमादिफलेभ्योऽथ नेया फलततिर्मुहुः ।
एकत्रयाद्योज संख्याभिर्भक्तेष्वेतेष्वनुक्रमात् ॥
ओजानां संयुतेस्त्यक्त्वा युग्मयोगं धनुर्भवेत् ।
दोःकोटयोरल्पमेवेह कल्पनीयमिह स्मृतम् ।
लब्धीनामवसानं स्यान्नान्यथापि मुहुः कृते ॥

Proof supplied by Jyeṣṭhadeva in his *Yuktibhāṣā* (1530)

Keralan series

From the *Tantrasamgraha-vyakhya*:

The product of the given Sine and the radius divided by the Cosine is the first result. From the first, [and then, second, third] etc., results obtain [successively] a sequence of results by taking repeatedly the square of the Sine as the multiplier and the square of the Cosine as the divisor. Divide [the above results] in order by the odd numbers one, three, etc. [to get the full sequence of terms]. From the sum of the odd terms, subtract the sum of the even terms. [The results] become the arc. In this connection, it is laid down that the [Sine of the] arc or [that of] its complement, which ever is smaller, should be taken here [as the given Sine]; otherwise, the terms obtained by the [above] repeated process will not tend to the vanishing magnitude.

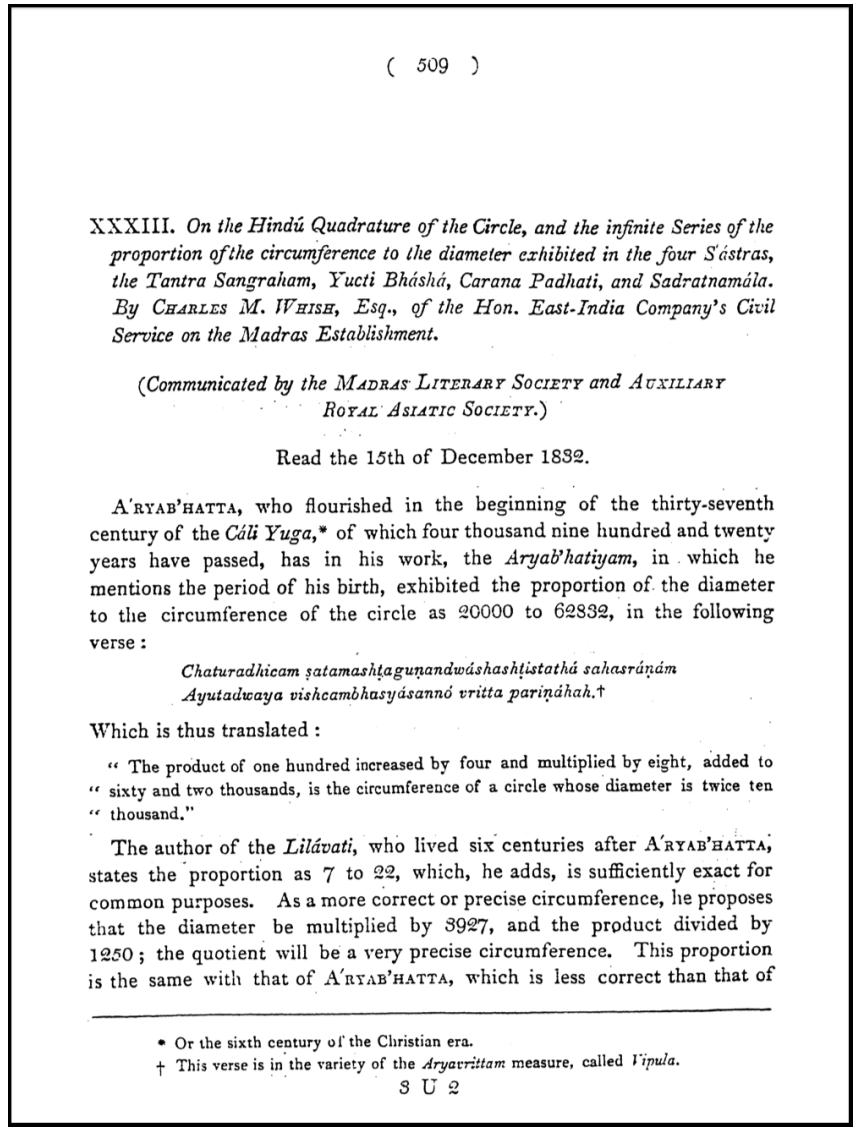
Modern interpretation:

$$R\theta = \frac{R(R\sin\theta)^1}{1(R\cos\theta)^1} - \frac{R(R\sin\theta)^3}{3(R\cos\theta)^3} + \frac{R(R\sin\theta)^5}{5(R\cos\theta)^5} - \cdots \qquad (R\sin\theta < R\cos\theta)$$

Keralan series

But these results were unknown in the West until the 1830s

As we will see, the series for arctan was reproduced independently in Scotland in the 1670s



XXXIII. *On the Hindú Quadrature of the Circle, and the infinite Series of the proportion of the circumference to the diameter exhibited in the four Sástras, the Tantra Sangraham, Yucti Bháshá, Carana Padhati, and Sadratnamála. By* CHARLES M. WHISH, *Esq., of the Hon. East-India Company's Civil Service on the Madras Establishment.*

(*Communicated by the* MADRAS LITERARY SOCIETY *and* AUXILIARY ROYAL ASIATIC SOCIETY.)

Read the 15th of December 1832.

A'RYAB'HATTA, who flourished in the beginning of the thirty-seventh century of the *Cáli Yuga*,* of which four thousand nine hundred and twenty years have passed, has in his work, the *Aryab'hatiyam*, in which he mentions the period of his birth, exhibited the proportion of the diameter to the circumference of the circle as 20000 to 62832, in the following verse :

Chaturadhicam śatamashṭaguṇandwáshashṭistathá sahasráṇám
Ayutadwaya vishcambhasyásannó vritta pariṇáhah.†

Which is thus translated :

" The product of one hundred increased by four and multiplied by eight, added to " sixty and two thousands, is the circumference of a circle whose diameter is twice ten " thousand."

The author of the *Lilávati*, who lived six centuries after A'RYAB'HATTA, states the proportion as 7 to 22, which, he adds, is sufficiently exact for common purposes. As a more correct or precise circumference, he proposes that the diameter be multiplied by 3927, and the product divided by 1250 ; the quotient will be a very precise circumference. This proportion is the same with that of A'RYAB'HATTA, which is less correct than that of

* Or the sixth century of the Christian era.

† This verse is in the variety of the *Aryavrittam* measure, called *Vipula*.

Keralan series

Warning!

It has sometimes been claimed that there must be a link between European and Keralan ideas about infinite series, because the same results occur in both places.

However, there is no documentary evidence of such a link.

In general,

$$\text{conceptual similarities} \neq \text{evidence of transmission}$$

Question: 'what are or should be the criteria for accepting a hypothesis of cross-cultural transmission as plausible, and are those criteria culturally dependent?' (Kim Plofker, *Mathematics in India*, Princeton University Press, 2009, p. 252)

Infinite series 1600–1900: an overview

Lecture VII:

- mid–late 17th century: many discoveries
- early 18th century: much progress
- later 18th century: doubts and questions

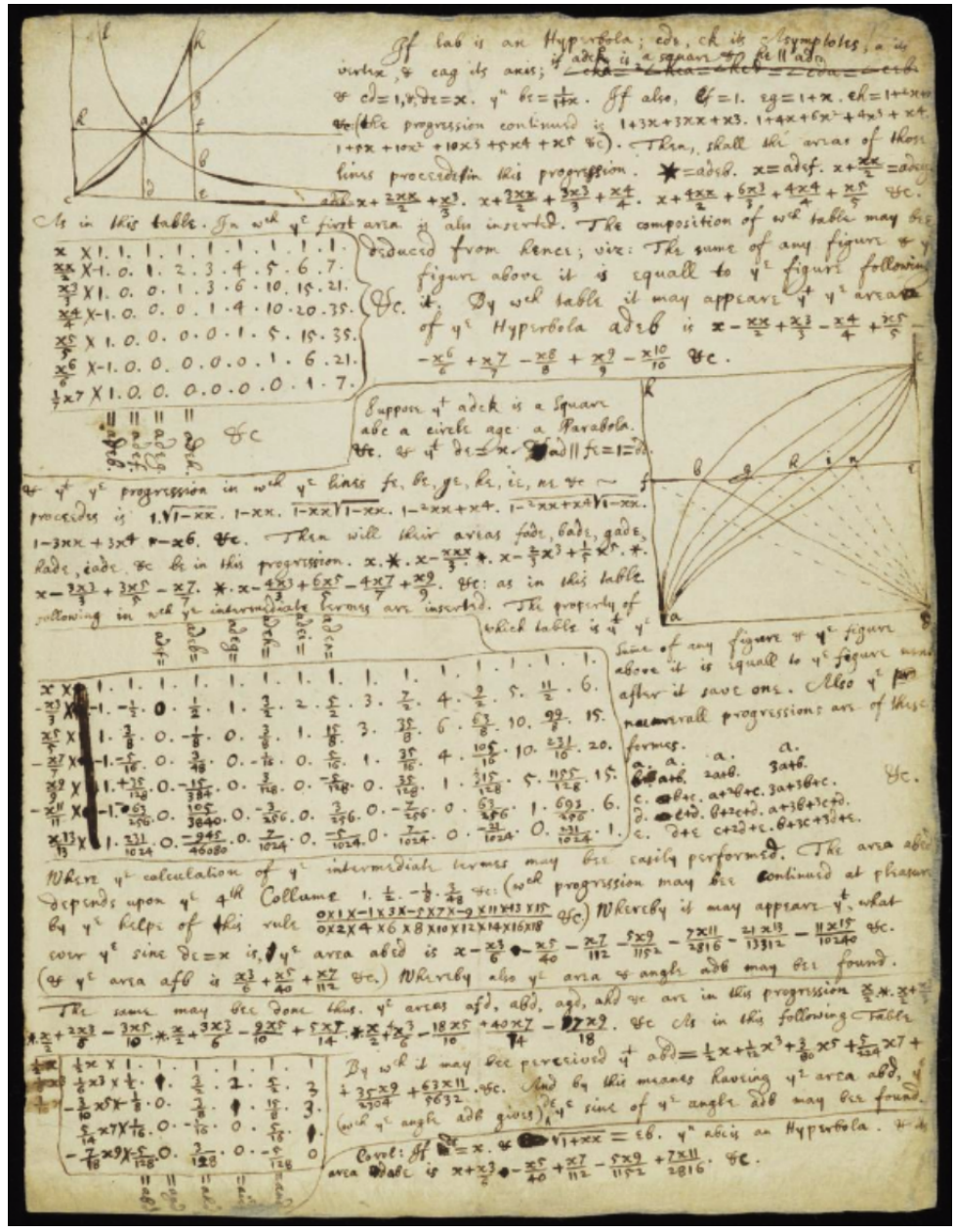
Lecture VIII:

- early 19th century: Fourier series
- early 19th century: convergence better understood

Newton and the general binomial theorem

CUL Add. MS 3958.3, f. 72

(See lecture IV)



Recall: Newton's integration of $(1+x)^{-1}$

	$(1+x)^{-1}$	$(1+x)^0$	$(1+x)^1$	$(1+x)^2$	$(1+x)^3$	$(1+x)^4$	$\cdots$
x	1	1	1	1	1	1	$\cdots$
$\frac{x^2}{2}$	-1	0	1	2	3	4	$\cdots$
$\frac{x^3}{3}$	1	0	0	1	3	6	$\cdots$
$\frac{x^4}{4}$	-1	0	0	0	1	4	$\cdots$
$\frac{x^5}{5}$	1	0	0	0	0	1	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

The entry in the row labelled $\frac{x^m}{m}$ and the column labelled $(1+x)^n$ is the coefficient of $\frac{x^m}{m}$ in $\int (1+x)^n dx$. (NB. Newton did not use the notation $\int (1+x)^n dx$.)

Newton's method of extrapolation

In fact, this method extends easily to any integer n

Newton's explanation:

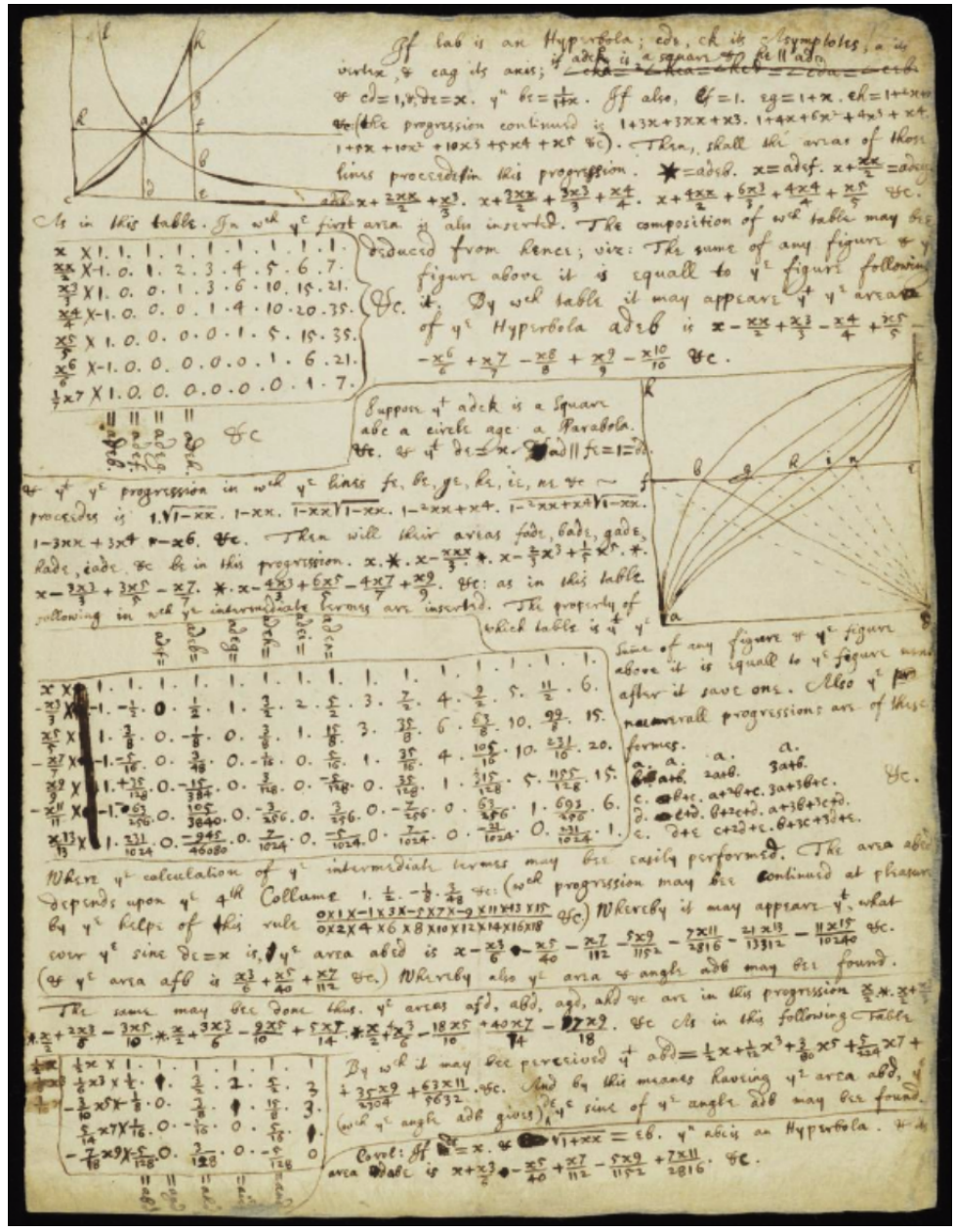
> *The property of which table is* y^{t} y^{e} *sum of any figure and* y^{e} *figure above it is equal to* y^{e} *figure next after it save one. Also* y^{e} *numerall progressions are of these forms.*

a	a	a	a	
b	$a+b$	$2a+b$	$3a+b$	
c	$b+c$	$a+2b+c$	$3a+3b+c$	&c.
d	$c+d$	$b+2c+d$	$a+3b+3c+d$	
e	$d+e$	$c+2d+e$	$b+3c+3d+e$	

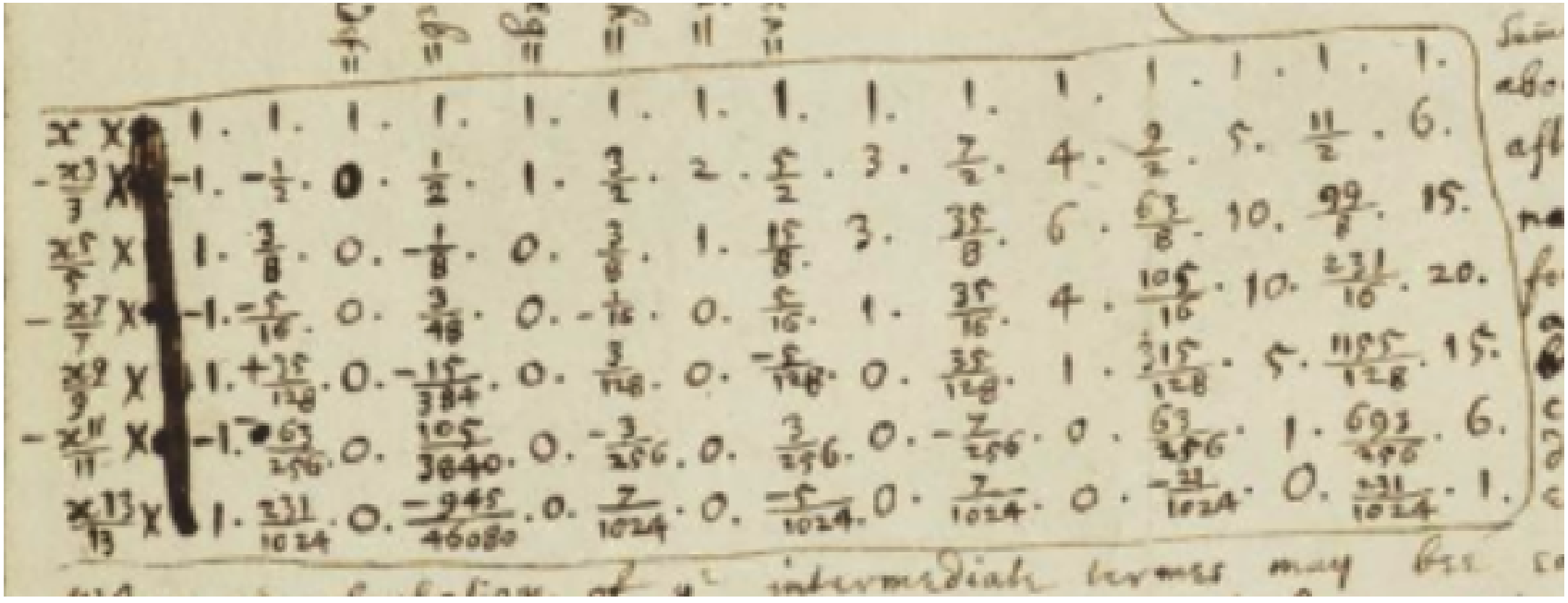
(See: *Mathematics emerging*, §8.1.1.)

Newton and the general binomial theorem

CUL Add. MS 3958.3, f. 72



Newton's method of interpolation



Newton's method of interpolation

	$(1-x^2)^{-1}$	$(1-x^2)^{-\frac{1}{2}}$	$(1-x^2)^0$	$(1-x^2)^{\frac{1}{2}}$	$(1-x^2)^1$	$(1-x^2)^{\frac{3}{2}}$	$(1-x^2)^2$	$\cdots$
x	1	1	1	1	1	1	1	$\cdots$
$-\frac{x^3}{3}$	-1	$-\frac{1}{2}$	0	$\frac{1}{2}$	1	$\frac{3}{2}$	2	$\cdots$
$\frac{x^5}{5}$	1	$\frac{3}{8}$	0	$-\frac{1}{8}$	0	$\frac{3}{8}$	1	$\cdots$
$-\frac{x^7}{7}$	-1	$-\frac{5}{16}$	0	$\frac{3}{48}$	0	$-\frac{1}{16}$	0	$\cdots$
$\frac{x^9}{9}$	1	$\frac{35}{128}$	0	$-\frac{15}{384}$	0	$\frac{3}{128}$	0	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

The entry in the row labelled $\pm\frac{x^m}{m}$ and the column labelled $(1-x^2)^n$ is the coefficient of $\pm\frac{x^m}{m}$ in $\int(1-x^2)^n dx$.

(NB: possible slips in the last two rows of the original table)

Newton's method of interpolation

Can fill in some initial values by other methods

Newton applied the formula

$$\binom{n}{k} = \frac{n(n-1)(n-2)\cdots(n-k+1)}{k!}$$

to fractional n, so that

$$\binom{1/2}{1} = \frac{1}{2}, \quad \binom{1/2}{2} = \frac{1/2(1/2-1)}{2!} = -\frac{1}{8}$$

and so on

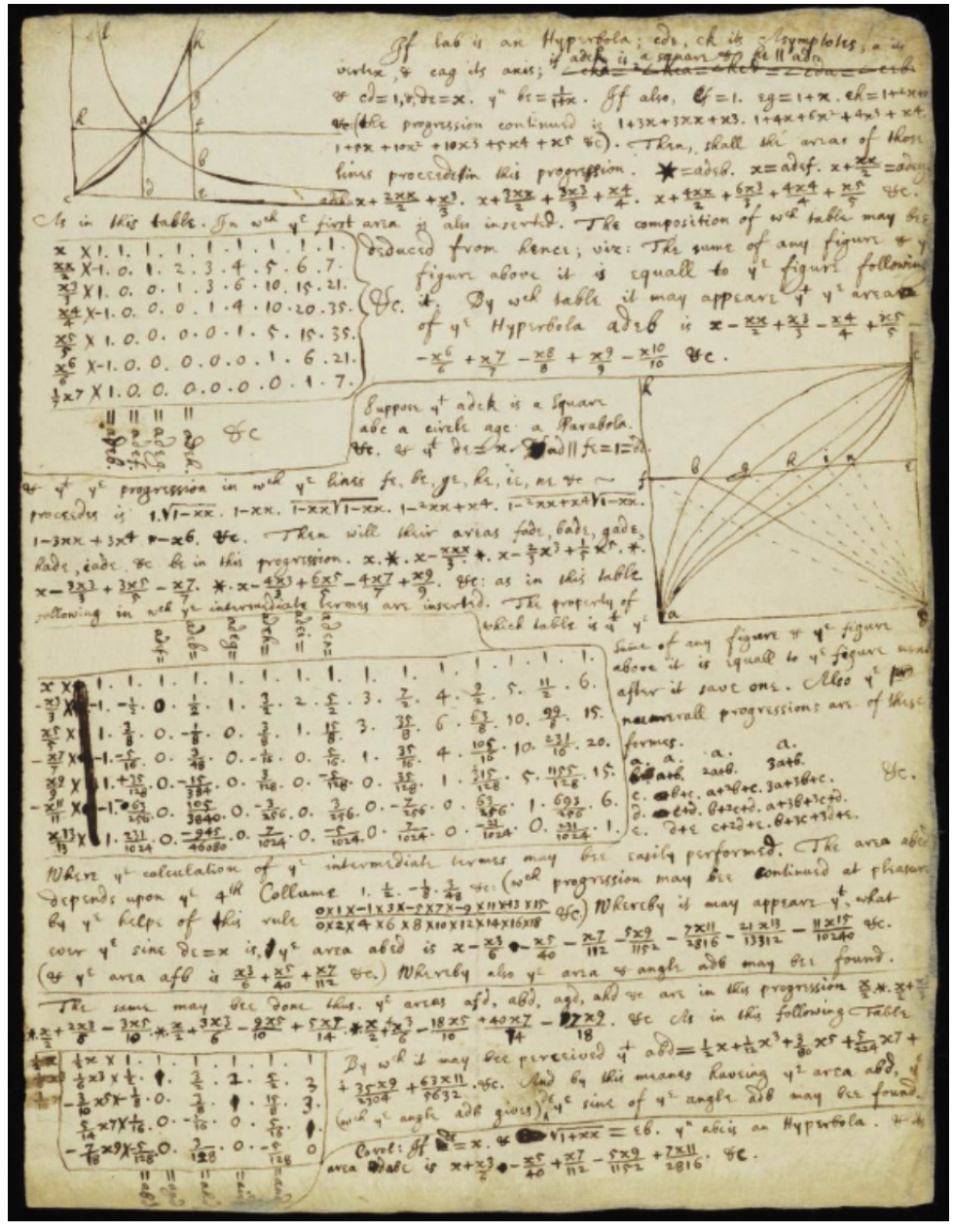
Newton's and the general binomial theorem

Newton went on to extend this method to other fractional powers, and also to $(a + bx)^n$, thereby convincing himself of the truth of the general binomial theorem — but this was not proved until the 19th century

On Newton and the binomial theorem, see
https://www.youtube.com/watch?v=xv_PWwdDWDk

One more table

The table at the bottom of the page gives the interpolations for $(1+x)^n$ for half-integer n



Further discoveries by Newton

By further interpolations and integrations (based on strong geometric intuition) Newton found further series for:

- $(1 + x)^{p/q}$
- log, antilog
- sin, tan, ... (NB: cosine was not yet much in use)
- arcsin, arctan, ...

(See: *Mathematics emerging*, §§8.1.2–8.1.3.)

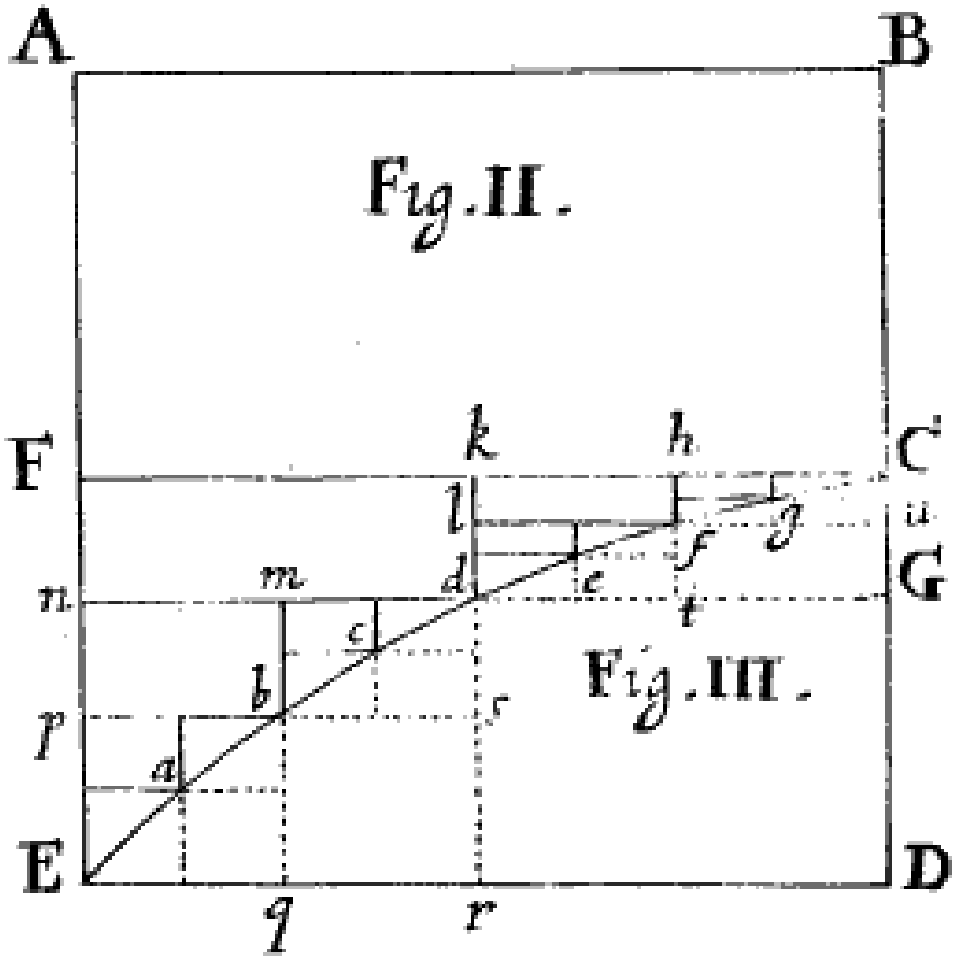
Newton on the move from finite to infinite series

> *And whatever common analysis performs by equations made up of a finite number of terms (whenever it may be possible), this method may always perform by infinite equations: in consequence, I have never hesitated to bestow on it also the name of analysis.*

(*De analysi*, 1669; Derek T. Whiteside, *The mathematical papers of Isaac Newton*, CUP, 1967–1981, vol. II, p. 241)

Other 17th-century discoveries (1a)

Brouncker, c. 1655, published 1668: area under the hyperbola given by $\frac{1}{1\times 2}+\frac{1}{3\times 4}+\frac{1}{5\times 6}+\cdots$



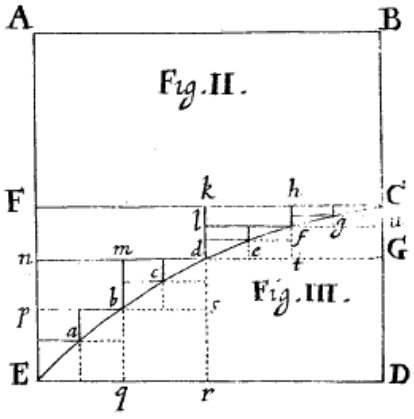
Other 17th-century discoveries (1b)

I say ABCdEA $= \frac{1}{1\times2} + \frac{1}{3\times4} + \frac{1}{5\times6} + \frac{1}{7\times8} + \frac{1}{9\times10}$ &c.

EdCDE $= \frac{1}{2\times3} + \frac{1}{4\times5} + \frac{1}{6\times7} + \frac{1}{8\times9} + \frac{1}{10\times11}$ &c.

EdCyE $= \frac{1}{2\times3\times4} + \frac{1}{4\times5\times6} + \frac{1}{6\times7\times8} + \frac{1}{8\times9\times10}$ &c.

} *in infinitum.*



And that therefore in the first series half the first term is greater than the sum of the two next, and half this sum of the second and third greater than the sum of the four next, and half the sum of those four greater than the sum of the next eight, &c. *in infinitum*. For $\frac{1}{2}\,dD = br + bn$; but $bn > fG$, therefore $\frac{1}{2}\,dD > br + fG$, &c. And in the *second* series half the first term is less then the sum of the two next, and half this sum less then the sum of the four next, &c. *in infinitum*.

That the first *series* are the *even* terms, *viz.* the 2d, 4th, 6th, 8th, 10th, &c. and the *second*, the *odd*, *viz.* the 1st, 3d, 5th, 7th, 9th, &c. of the following series, *viz.* $\frac{1}{1\times2} \cdot \frac{1}{3\times4} \cdot \frac{1}{5\times6} \cdot \frac{1}{7\times8} \cdot \frac{1}{9\times10}$ &c. *in infinitum* $= 1$. Whereof a being put for the number of terms taken at pleasure, $\frac{1}{a-1\ a}$ is the last, $\frac{a}{a+1}$ is the sum of all those terms from the beginning, and $\frac{1}{a+1}$ the sum of the rest to the end.

That $\frac{1}{4}$ of the first terme in the *third* series is less than the sum of the two next, and a quarter of this sum, less than the sum of the four next, and one fourth of this last sum less than the next eight, I thus demonstrate.

Let a = the 3d or last number of any term of the first Column, *viz.* of Divisors,

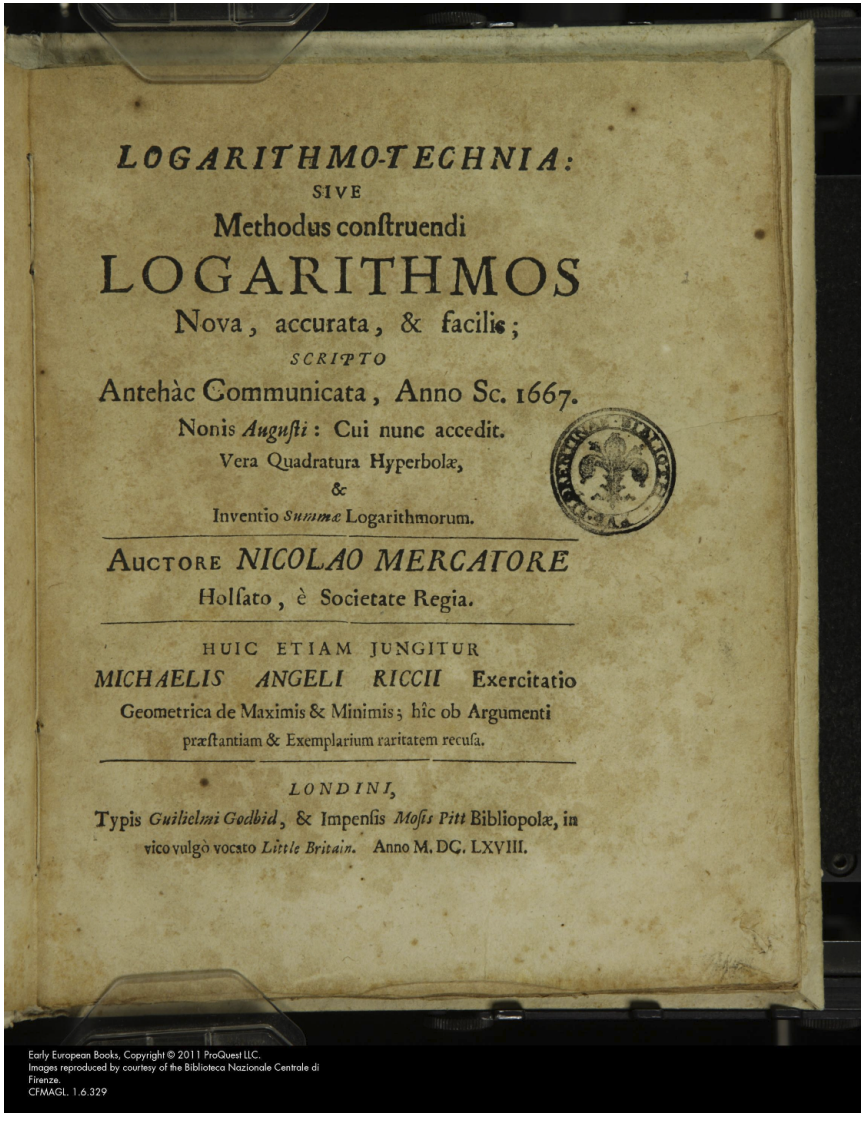
⋮

Other 17th-century discoveries (2)

Mercator's series (1668), found by long division:

$$\frac{1}{1+a} = 1 - a + aa - a^3 + a^4 \text{ (\&c.)}$$

Gives rise to series for log


LOGARITHMO-TECHNIA:
SIVE
Methodus conſtruendi
LOGARITHMOS
Nova, accurata, & facilis;
SCRIPTO
Antehàc Communicata, Anno Sc. 1667.
Nonis Auguſti: Cui nunc accedit.
Vera Quadratura Hyperbolæ,
&
Inventio Summæ Logarithmorum.
AUCTORE NICOLAO MERCATORE
Holſato, è Societate Regia.
HUIC ETIAM JUNGITUR
MICHAELIS ANGELI RICCII Exercitatio
Geometrica de Maximis & Minimis; hîc ob Argumenti
præſtantiam & Exemplarium raritatem recuſa.
LONDINI,
Typis Guilielmi Godbid, & Impenſis Moſis Pitt Bibliopolæ, in
vico vulgò vocato Little Britain. Anno M. DC. LXVIII.
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Other 17th-century discoveries (3)

James Gregory (1671):

- general binomial expansion
- series for tan, sec, and others, including

$$\theta = \tan\theta - \frac{1}{2}\tan^3\theta + \frac{1}{5}\tan^5\theta - \cdots$$

for $-\frac{\pi}{4} \leq \theta \leq \frac{\pi}{4}$

Gregory to Collins, 23rd November 1670:

> *I suppose these series I send here enclosed, may have some affinity with those inventions you advertise me that Mr. Newton had discovered.*

(On Gregory's work, see: *Mathematics emerging*, §8.1.4.)

Other 17th-century discoveries (4)

Gottfried Wilhelm Leibniz (1675):

The area of a circle with unit diameter is given by

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \&c.$$

The error in the sum is successively less than $\frac{1}{3}$, $\frac{1}{5}$, $\frac{1}{7}$, etc.

> *Therefore the series as a whole contains all approximations at once, or values greater than correct and less than correct: for according to how far it is understood to be continued, the error will be smaller than a given fraction, and therefore also less than any given quantity. Therefore the series as a whole expresses the exact value.*

(See: *Mathematics emerging*, §8.3.)

Series in the 17th century: 'convergence'

John Wallis (1656), *Arithmetica infinitorum*:

$$\square = \frac{4}{\pi} = \frac{3 \times 3 \times 5 \times 5 \times 7 \times 7 \times \cdots}{2 \times 4 \times 4 \times 6 \times 6 \times 8 \times \cdots}$$

(Determined that

$$\square > \sqrt{\frac{3}{2}}, \quad \square < \frac{3}{2}\sqrt{\frac{3}{4}}, \quad \square > \left(\frac{3 \times 3}{2 \times 4}\right)\sqrt{\frac{5}{4}},$$

and so on)

Brouncker (1668): grouping of terms

Leibniz (1675): 'alternating' series

Power series in the 17th century

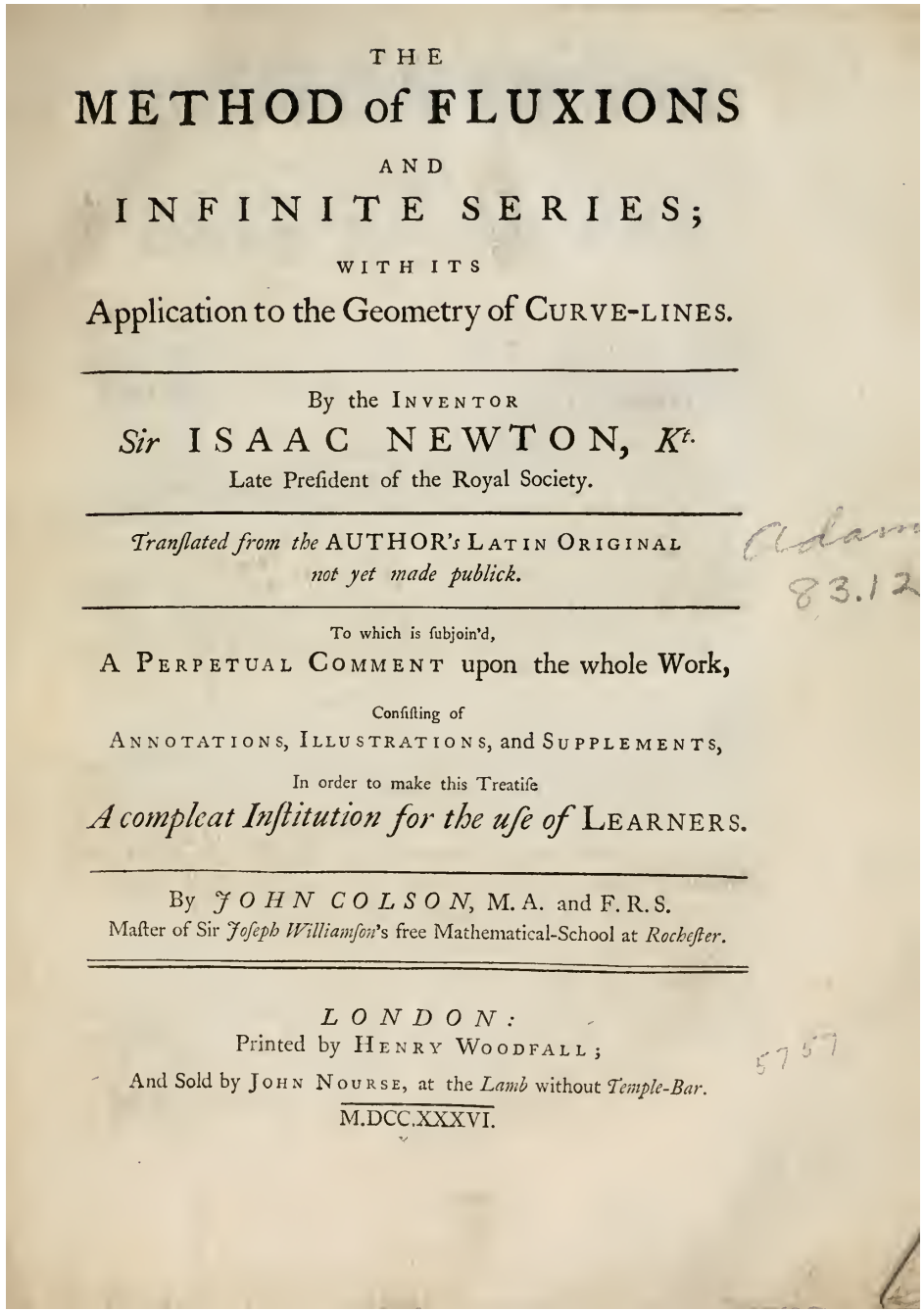
Power series (infinite polynomials):

- enabled term-by-term integration for difficult quadratures;
- helped establish sine, log, ... as 'functions' (transcendental);
- encouraged a move from geometric to algebraic descriptions;
- for Newton (and others) inextricably linked with calculus.

Power series rank with calculus as a major advance of the 17th century

Calculus and series combined

Newton's treatise of 1671, published 1736



THE

METHOD of FLUXIONS

AND

INFINITE SERIES;

WITH ITS

Application to the Geometry of CURVE-LINES.

By the INVENTOR

Sir ISAAC NEWTON, *Kt.*

Late Preſident of the Royal Society.

Tranſlated from the AUTHOR's LATIN ORIGINAL *not yet made publick.*

To which is ſubjoin'd,

A PERPETUAL COMMENT upon the whole Work,

Conſiſting of

ANNOTATIONS, ILLUSTRATIONS, and SUPPLEMENTS,

In order to make this Treatiſe

A compleat Inſtitution for the uſe of LEARNERS.

By *JOHN COLSON*, M.A. and F.R.S.

Maſter of Sir *Joſeph Williamſon*'s free Mathematical-School at *Rocheſter*.

LONDON:

Printed by HENRY WOODFALL;

And Sold by JOHN NOURSE, at the *Lamb* without *Temple-Bar*.

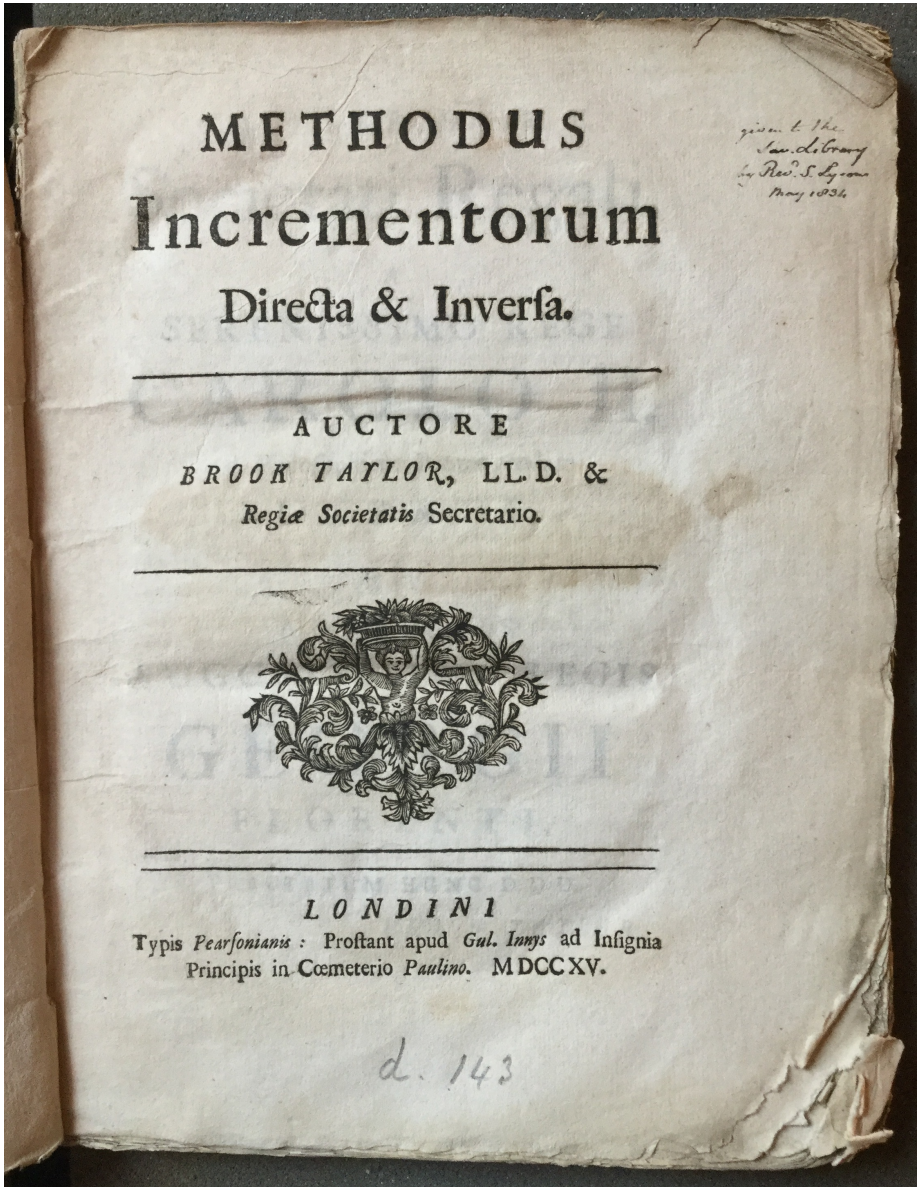
M.DCC.XXXVI.

Move on to the 18th century

Eighteenth century:

- as in 17th century, much progress;
- also many questions and doubts

Taylor series



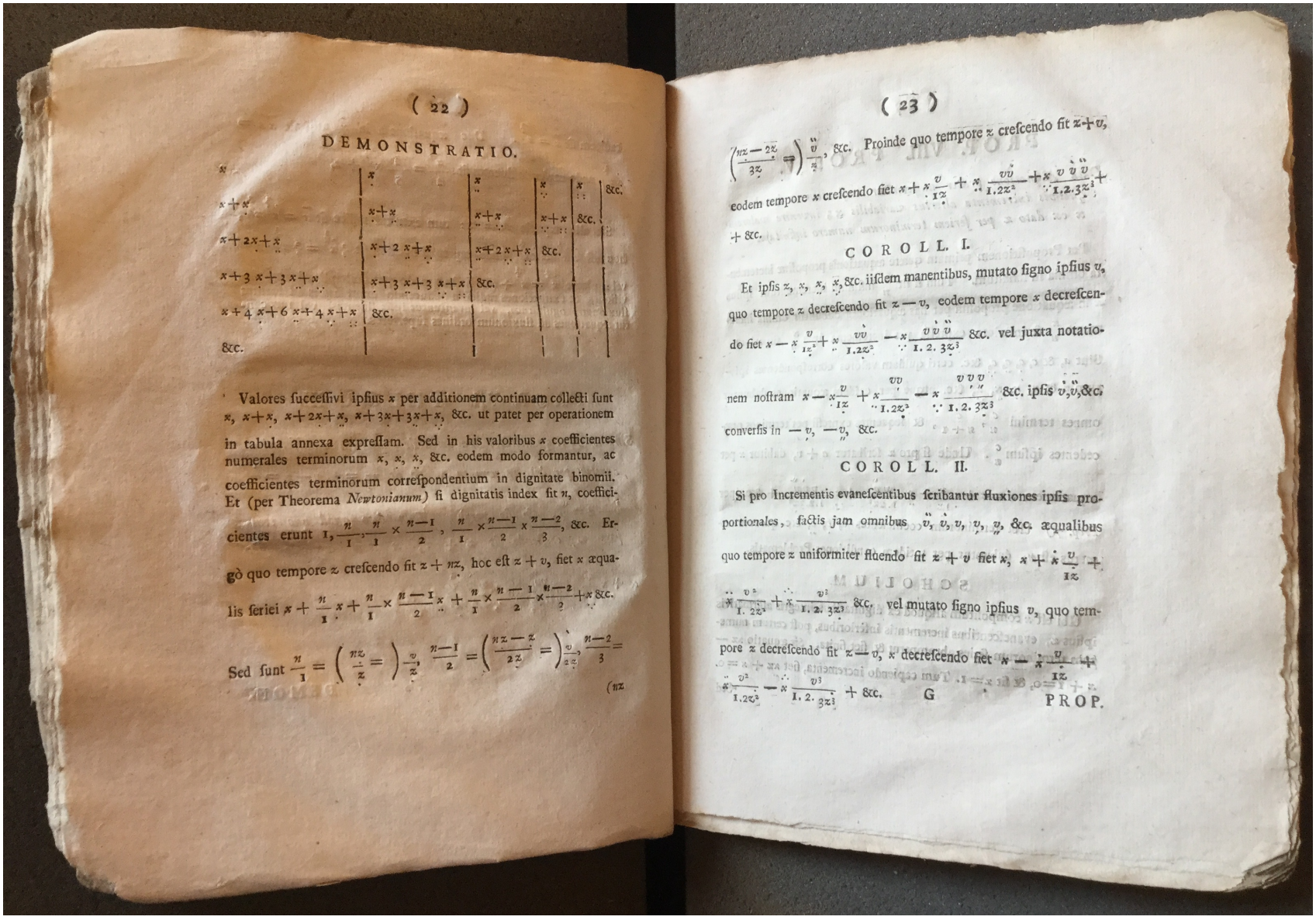
METHODUS
Incrementorum
Directa & Inverſa.

AUCTORE
BROOK TAYLOR, LL.D. &
Regiæ Societatis Secretario.

LONDINI
Typis *Pearſonianis* : Proſtant apud *Gul. Innys* ad Inſignia
Principis in Cœmeterio *Paulino*. MDCCXV.

Brook Taylor,
The method of direct and inverse increments (1715)

Taylor series



(22)

DEMONSTRATIO.

x	$\dot{x}$	$\ddot{x}$	$\dddot{x}$	$\ddddot{x}$	&c.
$x+\dot{x}$	$\dot{x}+\ddot{x}$	$\ddot{x}+\dddot{x}$	$\dddot{x}+\ddddot{x}$	&c.	
$x+2\dot{x}+\ddot{x}$	$\dot{x}+2\ddot{x}+\dddot{x}$	$\ddot{x}+2\dddot{x}+\ddddot{x}$	&c.		
$x+3\dot{x}+3\ddot{x}+\dddot{x}$	$\dot{x}+3\ddot{x}+3\dddot{x}+\ddddot{x}$	&c.			
$x+4\dot{x}+6\ddot{x}+4\dddot{x}+\ddddot{x}$	&c.				
&c.					

Valores ſucceſſivi ipſius x per additionem continuam collecti ſunt x, $x+\dot{x}$, $x+2\dot{x}+\ddot{x}$, $x+3\dot{x}+3\ddot{x}+\dddot{x}$, &c. ut patet per operationem in tabula annexa expreſſam. Sed in his valoribus x coefficientes numerales terminorum x, $\dot{x}$, $\ddot{x}$, &c. eodem modo formantur, ac coefficientes terminorum correſpondentium in dignitate binomii. Et (per Theorema *Newtonianum*) ſi dignitatis index ſit n, coefficientes erunt $1, \frac{n}{1}, \frac{n}{1}\times\frac{n-1}{2}, \frac{n}{1}\times\frac{n-1}{2}\times\frac{n-2}{3}$, &c. Ergò quo tempore z creſcendo fit $z+nz$, hoc eſt $z+v$, fiet x æqualis ſeriei $x+\frac{n}{1}\dot{x}+\frac{n}{1}\times\frac{n-1}{2}\ddot{x}+\frac{n}{1}\times\frac{n-1}{2}\times\frac{n-2}{3}\dddot{x}$ &c.

Sed ſunt $\frac{n}{1}=\left(\frac{nz}{\dot{z}}=\right)\frac{v}{\dot{z}}$, $\frac{n-1}{2}=\left(\frac{nz-\dot{z}}{2\dot{z}}=\right)\frac{v}{2\dot{z}}$, $\frac{n-2}{3}=$ (nz

(23)

$\left(\frac{nz-2\dot{z}}{3\dot{z}}=\right)\frac{\ddot{v}}{\dot{z}}$, &c. Proinde quo tempore z creſcendo fit $z+v$, eodem tempore x creſcendo fiet $x+\dot{x}\frac{v}{1\dot{z}}+\ddot{x}\frac{v\dot{v}}{1.2\dot{z}^2}+\dddot{x}\frac{v\dot{v}\ddot{v}}{1.2.3\dot{z}^3}+$ + &c.

COROLL. I.

Et ipſis z, $\dot{x}$, $\ddot{x}$, $\dddot{x}$, &c. iiſdem manentibus, mutato ſigno ipſius v, quo tempore z decreſcendo fit $z-v$, eodem tempore x decreſcendo fiet $x-\dot{x}\frac{v}{1\dot{z}}+\ddot{x}\frac{v\dot{v}}{1.2\dot{z}^2}-\dddot{x}\frac{v\dot{v}\ddot{v}}{1.2.3\dot{z}^3}$ &c. vel juxta notationem noſtram $x-\dot{x}\frac{v}{1\dot{z}}+\ddot{x}\frac{v\dot{v}}{1.2\dot{z}^2}-\dddot{x}\frac{v\dot{v}\ddot{v}}{1.2.3\dot{z}^3}$ &c. ipſis $\dot{v},\ddot{v},$&c. converſis in $-v$, $-\dot{v}$, &c.

COROLL. II.

Si pro Incrementis evaneſcentibus ſcribantur fluxiones ipſis proportionales, factis jam omnibus $\ddot{v}$, $\dot{v}$, v, $\dot{v}$, $\ddot{v}$, &c. æqualibus quo tempore z uniformiter fluendo fit $z+v$ fiet x, $x+\dot{x}\frac{v}{1\dot{z}}+\ddot{x}\frac{v^2}{1.2\dot{z}^2}+\dddot{x}\frac{v^3}{1.2.3\dot{z}^3}$ &c. vel mutato ſigno ipſius v, quo tempore z decreſcendo fit $z-v$, x decreſcendo fiet $x-\dot{x}\frac{v}{1\dot{z}}+\ddot{x}\frac{v^2}{1.2\dot{z}^2}-\dddot{x}\frac{v^3}{1.2.3\dot{z}^3}$ + &c.

(See: *Mathematics emerging*, §8.2.1.)

Taylor series

Taylor denoted a small change in x by $\dot{x}$ (our δx), a small change in $\dot{x}$ by $\ddot{x}$ (our $\delta(\delta x)$), and so on

Dependent variable x; independent variable z increases uniformly with time

x increases to $x + \delta x$ in time δt; after a further interval of δt, x has become $x + \delta x + \delta(x + \delta x) = x + 2\delta x + \delta(\delta x)$; continuing:

$$x + \frac{n}{1}\delta x + \frac{n(n-1)}{1 \cdot 2}\delta(\delta x) + \frac{n(n-1)(n-2)}{1 \cdot 2 \cdot 3}\delta(\delta(\delta x)) + \cdots$$

$$= x + \delta x \frac{n\delta z}{1\delta z} + \delta(\delta x)\frac{n\delta z(n-1)\delta z}{1 \cdot 2 \cdot (\delta z)^2} + \delta(\delta(\delta x))\frac{n\delta z(n-1)\delta z(n-2)\delta z}{1 \cdot 2 \cdot 3(\delta z)^3} + \cdots$$

Taylor series

$$x + \delta x \frac{n\delta z}{1\delta z} + \delta(\delta x) \frac{n\delta z(n-1)\delta z}{1 \cdot 2 \cdot (\delta z)^2} + \delta(\delta(\delta x)) \frac{n\delta z(n-1)\delta z(n-2)\delta z}{1 \cdot 2 \cdot 3(\delta z)^3} + \cdots$$

Assumptions:

- $(n-k)\delta z \approx n\delta z$, since δz is small, so replace each $(n-k)\delta z$ by v, a constant
- $\delta x \propto \dot{x}$ and $\delta z \propto \dot{z}$, so in each case the former can be replaced by the latter

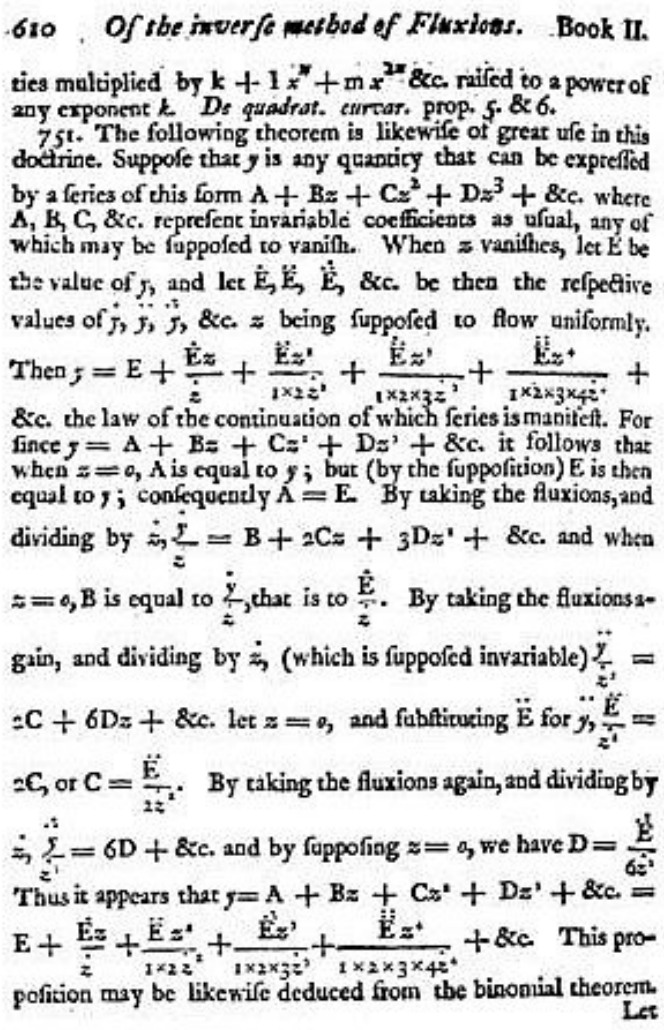
In essence (in modern terms): $\frac{\delta x}{\delta z} \to \frac{dx}{dz}$, $\frac{\delta(\delta x)}{(\delta z)^2} \to \frac{d^2x}{dz^2}$, and so on

Again in modern terms, we arrive at:

$$x + \frac{dx}{dz} v + \frac{d^2x}{dz^2} \frac{v^2}{1 \cdot 2} + \frac{d^3x}{dz^3} \frac{v^3}{1 \cdot 2 \cdot 3} + \cdots$$

Cf. Taylor's notation in *Mathematics Emerging*, §8.1.2

Maclaurin's *Treatise of fluxions*, vol. II, p. 610



610 *Of the inverſe method of Fluxions.* Book II.

ties multiplied by $k + l x^n + m x^{2n}$ &c. raiſed to a power of any exponent k. *De quadrat. curvar.* prop. 5. & 6.

751. The following theorem is likewiſe of great uſe in this doctrine. Suppoſe that y is any quantity that can be expreſſed by a ſeries of this form $A + Bz + Cz^2 + Dz^3 +$ &c. where A, B, C, &c. repreſent invariable coefficients as uſual, any of which may be ſuppoſed to vaniſh. When z vaniſhes, let E be the value of y, and let $\dot{E}, \ddot{E}, \dddot{E}$, &c. be then the reſpective values of $\dot{y}, \ddot{y}, \dddot{y}$, &c. z being ſuppoſed to flow uniformly. Then $y = E + \frac{\dot{E}z}{\dot{z}} + \frac{\ddot{E}z^2}{1\times 2\dot{z}^2} + \frac{\dddot{E}z^3}{1\times 2\times 3\dot{z}^3} +$ &c. the law of the continuation of which ſeries is manifeſt. For ſince $y = A + Bz + Cz^2 + Dz^3 +$ &c. it follows that when $z = o$, A is equal to y; but (by the ſuppoſition) E is then equal to y; conſequently $A = E$. By taking the fluxions, and dividing by $\dot{z}$, $\frac{\dot{y}}{\dot{z}} = B + 2Cz + 3Dz^2 +$ &c. and when $z = o$, B is equal to $\frac{\dot{y}}{\dot{z}}$, that is to $\frac{\dot{E}}{\dot{z}}$. By taking the fluxions again, and dividing by $\dot{z}$, (which is ſuppoſed invariable) $\frac{\ddot{y}}{\dot{z}^2} = 2C + 6Dz +$ &c. let $z = o$, and ſubſtituting $\ddot{E}$ for $\ddot{y}$, $\frac{\ddot{E}}{\dot{z}^2} = 2C$, or $C = \frac{\ddot{E}}{2\dot{z}^2}$. By taking the fluxions again, and dividing by $\dot{z}$, $\frac{\dddot{y}}{\dot{z}^3} = 6D +$ &c. and by ſuppoſing $z = o$, we have $D = \frac{\dddot{E}}{6\dot{z}^3}$. Thus it appears that $y = A + Bz + Cz^2 + Dz^3 +$ &c. $= E + \frac{\dot{E}z}{\dot{z}} + \frac{\ddot{E}z^2}{1\times 2\dot{z}^2} + \frac{\dddot{E}z^3}{1\times 2\times 3\dot{z}^3} + \frac{\ddddot{E}z^4}{1\times 2\times 3\times 4\dot{z}^4} +$ &c. This propoſition may be likewiſe deduced from the binomial theorem. Let

Suppose that y can be expressed as $A + Bz + Cz^2 + Dz^3 + \cdots$

When z vanishes, $y = E$, $\dot{y} = \dot{E}$, $\ddot{y} = \ddot{E}$, $\dddot{y} = \dddot{E}$, and so on

z is assumed to flow uniformly, so that $\dot{z} = \text{const}$

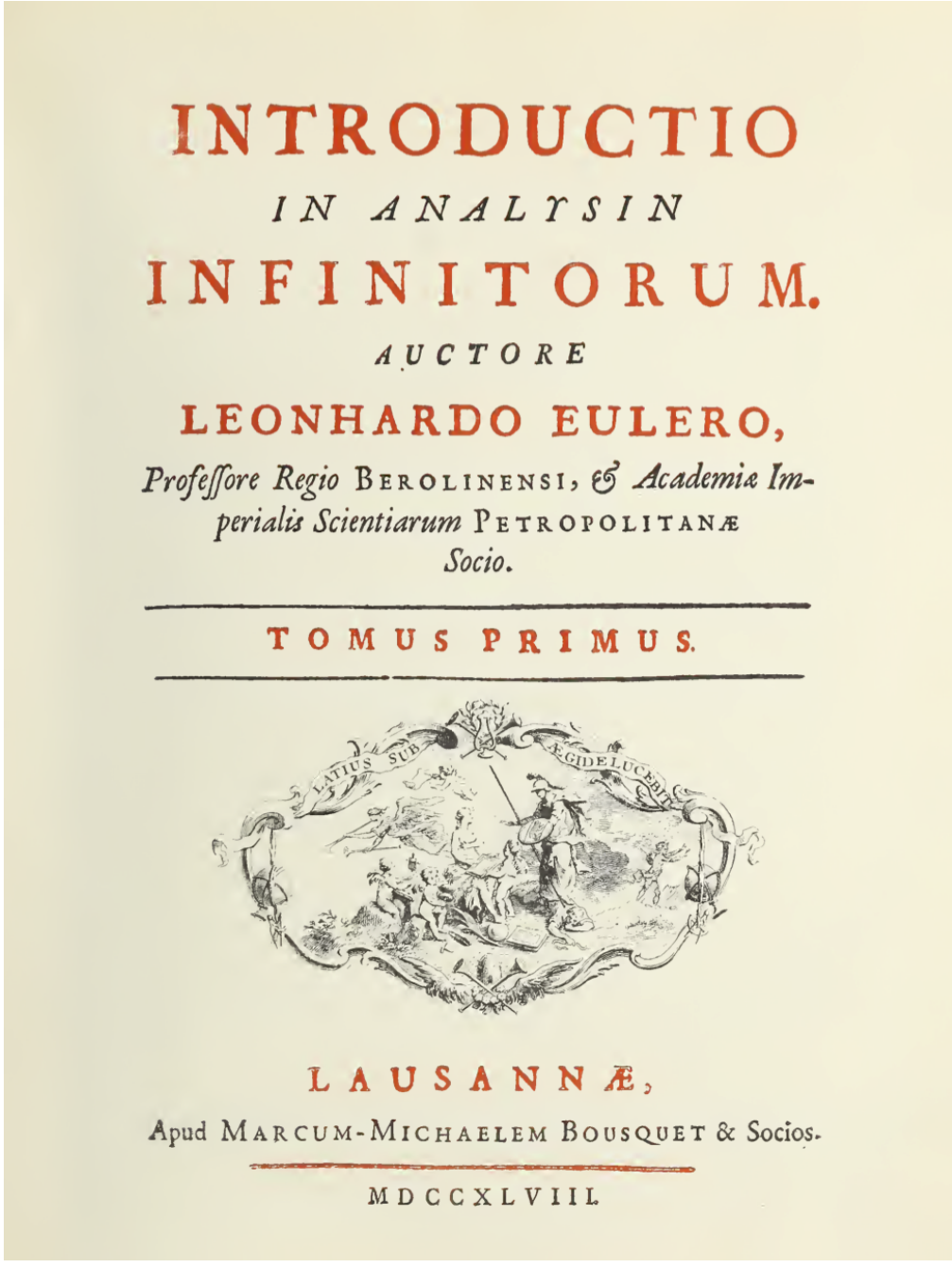
By repeatedly taking fluxions, we may calculate in turn: $A = E$, $B = \dot{E}\dot{z}$, $C = \frac{\ddot{E}}{2\ddot{z}^2}$, $D = \frac{\dddot{E}}{6\dot{z}^3}$, etc.

"the law of the continuation of [the] series is manifest"

(*Mathematics emerging*, §8.2.2.)

Euler's *Introductio*

Leonhard Euler, *Introduction to analysis of the infinite* (1748)


INTRODUCTIO

IN ANALYSIN

INFINITORUM.

AUCTORE

LEONHARDO EULERO,

Professore Regio BEROLINENSI, & Academiæ Imperialis Scientiarum PETROPOLITANÆ Socio.

TOMUS PRIMUS.

LATIUS SUB Æ GIDE LUCEBIT

LAUSANNÆ,

Apud MARCUM-MICHAELEM BOUSQUET & Socios.

MDCCXLVIII.

Euler's *Introductio*

Incorporated power series into the definition of a function:

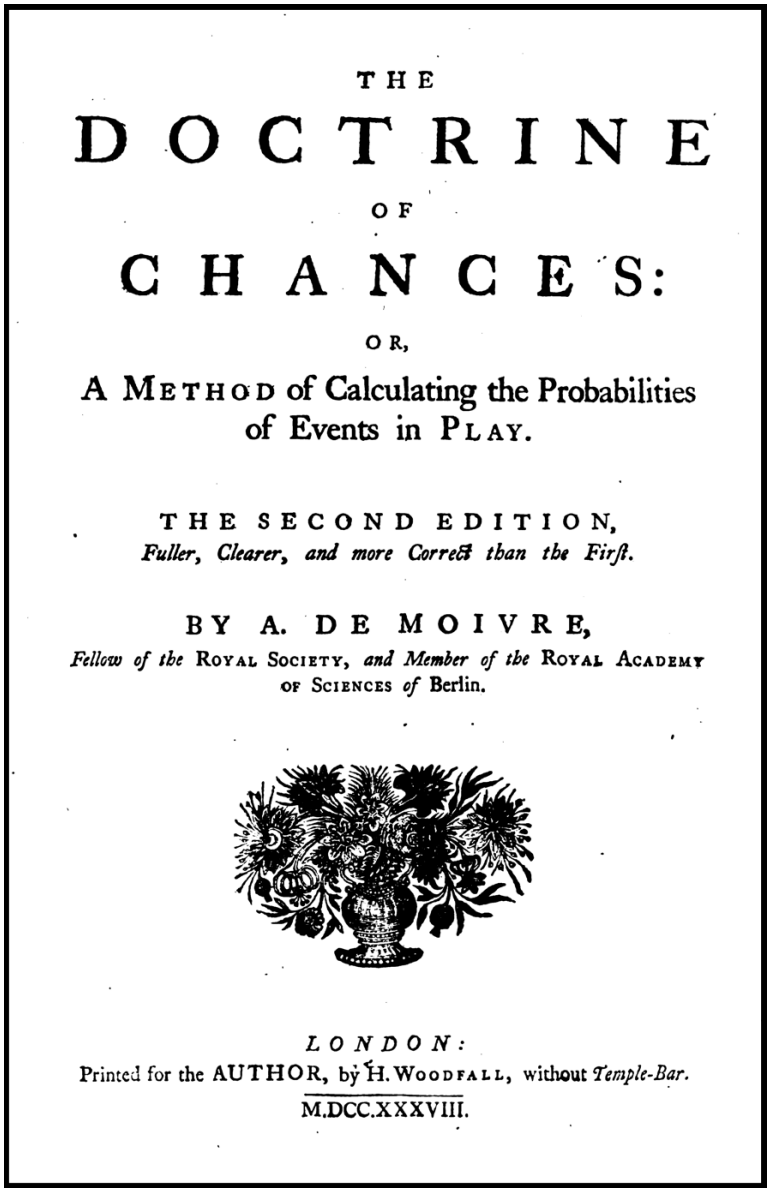
> *A* function *of a variable quantity is an analytic expression composed in any way whatsoever of the variable quantity and numbers or constant quantities.*

Euler derived series for sine, cosine, exp, log, etc.;

he also discovered relationships between them, for example:

$$\cos v = \frac{1}{2}(e^{iv} + e^{-iv})$$

An application of series



THE

DOCTRINE

OF

CHANCES:

OR,

A METHOD of Calculating the Probabilities of Events in PLAY.

THE SECOND EDITION,
Fuller, Clearer, and more Correct than the First.

BY A. DE MOIVRE,
Fellow of the ROYAL SOCIETY, *and Member of the* ROYAL ACADEMY OF SCIENCES *of* Berlin.

LONDON:
Printed for the AUTHOR, by H. WOODFALL, without *Temple-Bar.*
M.DCC.XXXVIII.

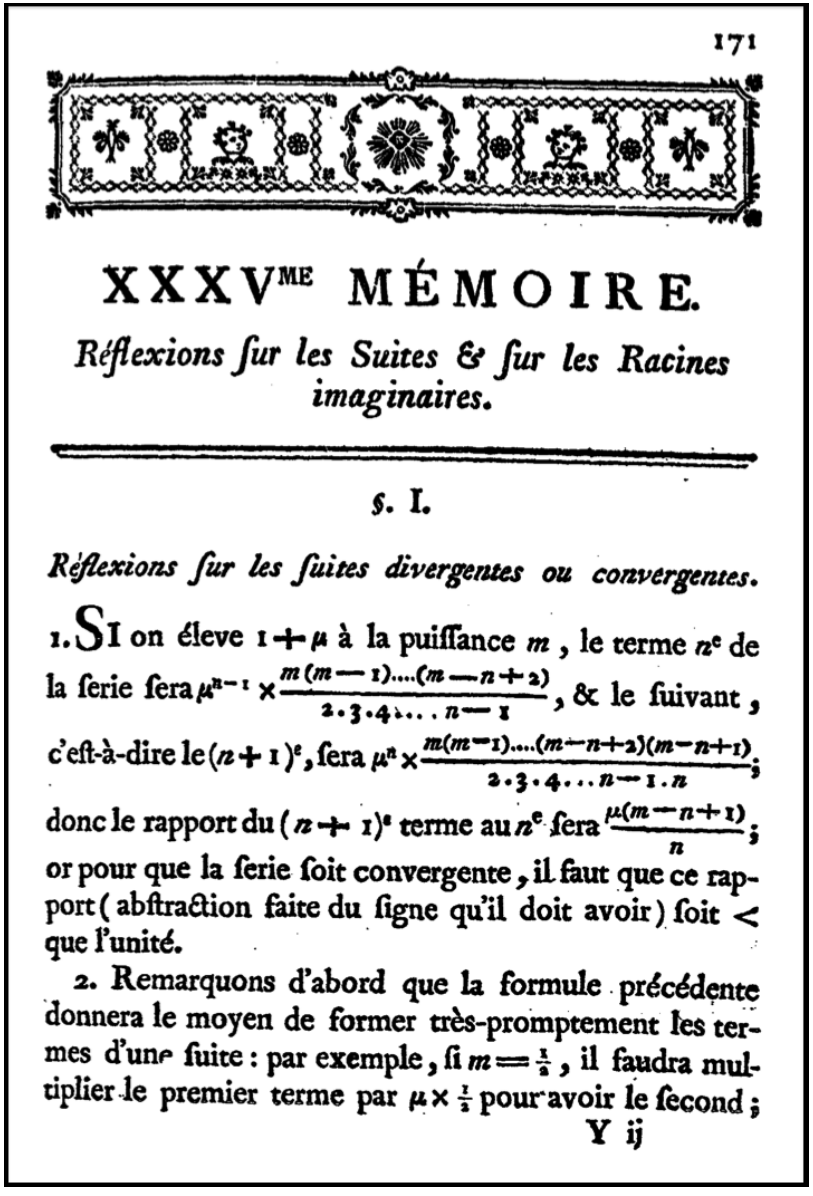
Abraham de Moivre posed this problem about confidence intervals:

What are the Odds that after a certain number of Experiments have been made concerning the happening or failing of Events, the Accidents of Contingency will not afterwards vary from those of Observation beyond certain Limits?

His answer involved clever (but non-rigorous) summation and manipulation of infinite series.

(*Mathematics emerging*, §7.1.3.)

Doubts



171

XXXV^ME MÉMOIRE.

Réflexions sur les Suites & sur les Racines imaginaires.

§. I.

Réflexions sur les suites divergentes ou convergentes.

1. SI on éleve $1+\mu$ à la puiſſance m, le terme n^e de la ſerie ſera $\mu^{n-1} \times \frac{m(m-1)\ldots(m-n+2)}{2\cdot 3\cdot 4\ldots n-1}$, & le ſuivant, c'eſt-à-dire le $(n+1)^e$, ſera $\mu^n \times \frac{m(m-1)\ldots(m-n+2)(m-n+1)}{2\cdot 3\cdot 4\ldots n-1\cdot n}$; donc le rapport du $(n+1)^e$ terme au n^e ſera $\frac{\mu(m-n+1)}{n}$; or pour que la ſerie ſoit convergente, il faut que ce rapport (abſtraction faite du ſigne qu'il doit avoir) ſoit $<$ que l'unité.

2. Remarquons d'abord que la formule précédente donnera le moyen de former très-promptement les termes d'une ſuite : par exemple, ſi $m = \frac{1}{2}$, il faudra multiplier le premier terme par $\mu \times \frac{1}{2}$ pour avoir le ſecond ;

Y ij

D'Alembert, 1761:

... all reasoning and calculation based on series that do not converge, or that one may suppose not to, always seems to me extremely suspect, even when the results of this reasoning agree with truths known in other ways.

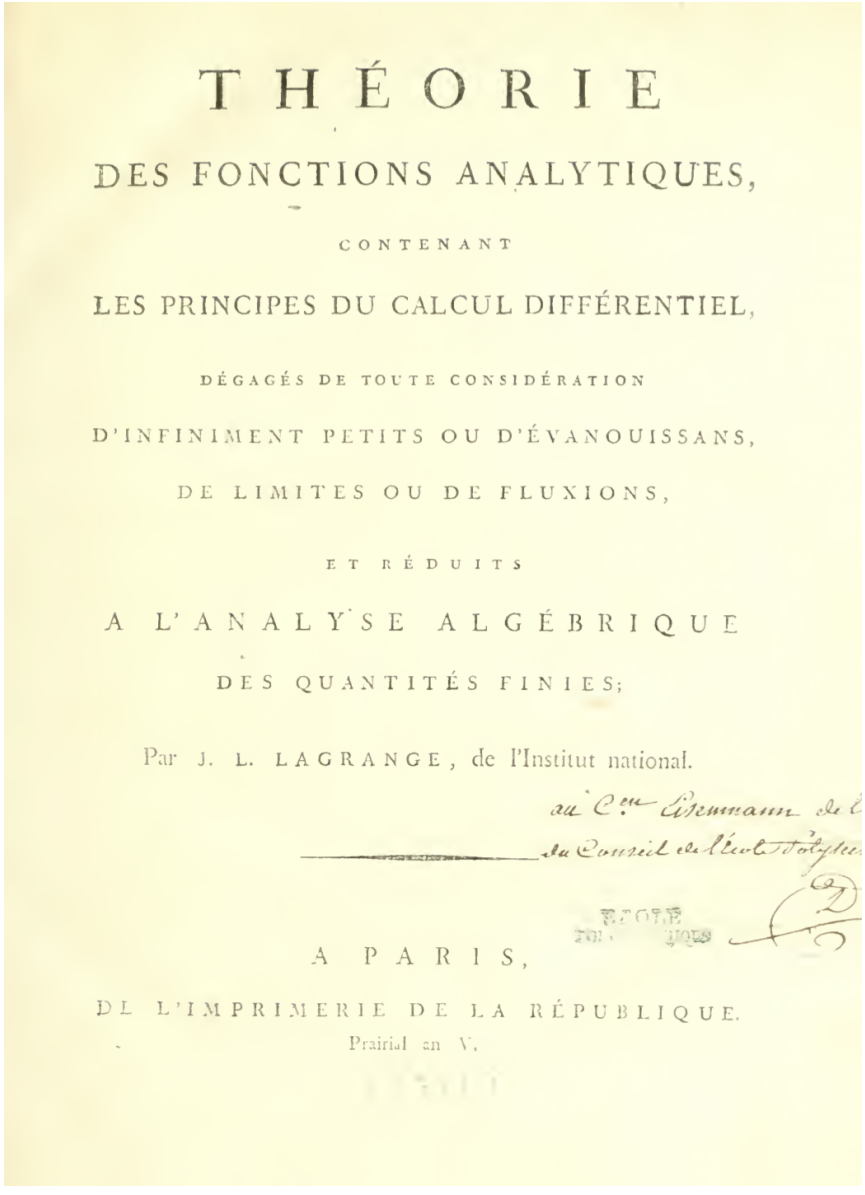
Introduced, without proof, what came to be known (in a more general setting) as d'Alembert's ratio test.

(See: *Mathematics emerging*, §8.3.1.)

Lagrange's use of series

J.-L. Lagrange, *Théorie des fonctions analytiques* (1797)

Lagrange's use of series: an attempt to liberate calculus from infinitely small quantities (essentially by treating only those functions that may be described by power series)



THÉORIE

DES FONCTIONS ANALYTIQUES,

CONTENANT

LES PRINCIPES DU CALCUL DIFFÉRENTIEL,

DÉGAGÉS DE TOUTE CONSIDÉRATION

D'INFINIMENT PETITS OU D'ÉVANOUISSANS,

DE LIMITES OU DE FLUXIONS,

ET RÉDUITS

A L'ANALYSE ALGÉBRIQUE

DES QUANTITÉS FINIES;

Par J. L. LAGRANGE, de l'Institut national.

A PARIS,

DE L'IMPRIMERIE DE LA RÉPUBLIQUE.

Prairial an V.

Lagrange and convergence

> ... [*one needs*] *a way of stopping the expansion of the series at any term one wants and of estimating the value of the remainder of the series.*
>
> *This problem, one of the most important in the theory of series, has not yet been resolved in a general way*

Lagrange found bounds for the 'remainder' ...

and applied his findings to the binomial series ...

thus proving what Newton had taken for granted

(See: *Mathematics emerging*, §8.3.2.)

BO1 History of Mathematics
Lecture VIII
Establishing rigorous thinking in analysis

MT 2022 Week 4

Summary

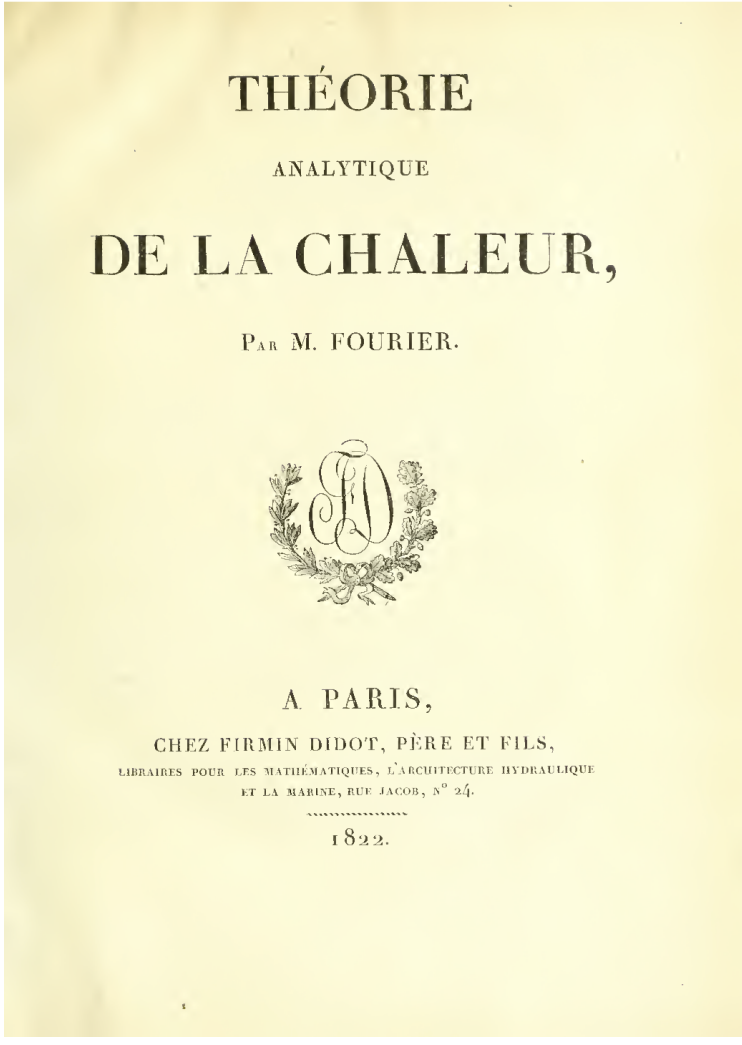
- French institutions
- Fourier series
- Early-19th-century rigour
- Limits, continuity, differentiability
- Mathematics of small quantities
- The baton passes from France to Germany

France at the turn of the 19th century

French revolution (1789) led to

- the establishment and prestige of the Grandes écoles
- a new concentration of mathematical talent in Paris: Lagrange, Laplace, Ampère, Fourier, Legendre, Poisson, Cauchy, ...
- a wealth of personal interactions
- a major new journal: *Journal de l'École polytechnique*
- and a new focus on rigour

Fourier series



THÉORIE

ANALYTIQUE

DE LA CHALEUR,

PAR M. FOURIER.

A PARIS,

CHEZ FIRMIN DIDOT, PÈRE ET FILS,

LIBRAIRES POUR LES MATHÉMATIQUES, L'ARCHITECTURE HYDRAULIQUE ET LA MARINE, RUE JACOB, N° 24.

1822.

Joseph Fourier, *Analytic theory of heat*, 1822

See: Bernard Maurey, 'Fourier, one man, several lives', *European Mathematical Society Newsletter*, no. 113 (Sept 2019), 8–20

Fourier series

Suppose that $\phi(x) = a \sin x + b \sin 2x + c \sin 3x + \cdots$

and also that $\phi(x) = x\phi'(0) + \frac{1}{2}x^2\phi''(0) + \frac{1}{6}x^3\phi'''(0) + \cdots$

After many pages of calculations, multiplying and comparing power series, Fourier found that the coefficient of $\sin nx$ must be

$$\frac{2}{\pi} \int_0^\pi \phi(x) \sin nx \, dx$$

Fourier's derivation was based on 'naive' manipulations of infinite series. It was ingenious but non-rigorous and shaky (see: *Mathematics emerging*, §8.4.1).

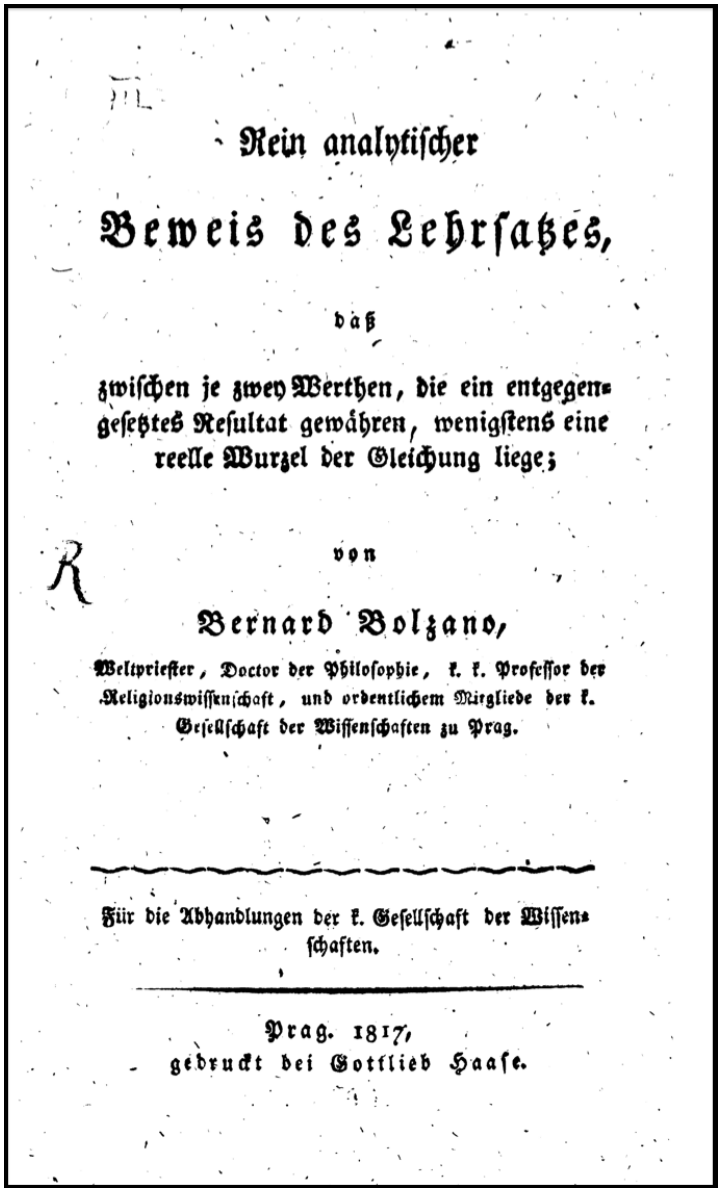
BUT it led to profound results

Establishing rigour

The development of 'rigour':

- Cauchy sequences
- continuity
- limits
- differentiability
- ϵ, δ notation

Cauchy sequences: Bolzano (1817)



Rein analytiſcher

Beweis des Lehrſatzes,

daß

zwiſchen je zwey Werthen, die ein entgegengeſetztes Reſultat gewähren, wenigſtens eine reelle Wurzel der Gleichung liege;

von

Bernard Bolzano,

Weltprieſter, Doctor der Philoſophie, k. k. Profeſſor der Religionswiſſenſchaft, und ordentlichem Mitgliede der k. Geſellſchaft der Wiſſenſchaften zu Prag.

Für die Abhandlungen der k. Geſellſchaft der Wiſſenſchaften.

Prag. 1817,
gedruckt bei Gottlieb Haaſe.

Bernard Bolzano, *Purely analytic proof of the theorem that between any two values which give opposite values lies at least one real root of the equation*, 1817

Cauchy sequences: Bolzano (1817)

Began with a discussion of previous proofs of the Intermediate Value Theorem (by "mathematicians of great repute")

> *The most common kind of proof depends on a truth borrowed from geometry, namely, that every continuous line of simple curvature of which the ordinates are first positive and then negative (or conversely) must necessarily intersect the x-axis somewhere at a point that lies in between those ordinates. There is certainly no question concerning the correctness, nor indeed the obviousness, of this geometrical propositon. But it is clear that it is an intolerable offense against correct method to derive truths of pure (or general) mathematics (i.e., arithmetic, algebra, analysis) from considerations which belong to a merely applied (or special) part, namely, geometry.*

Cauchy sequences: Bolzano (1817)

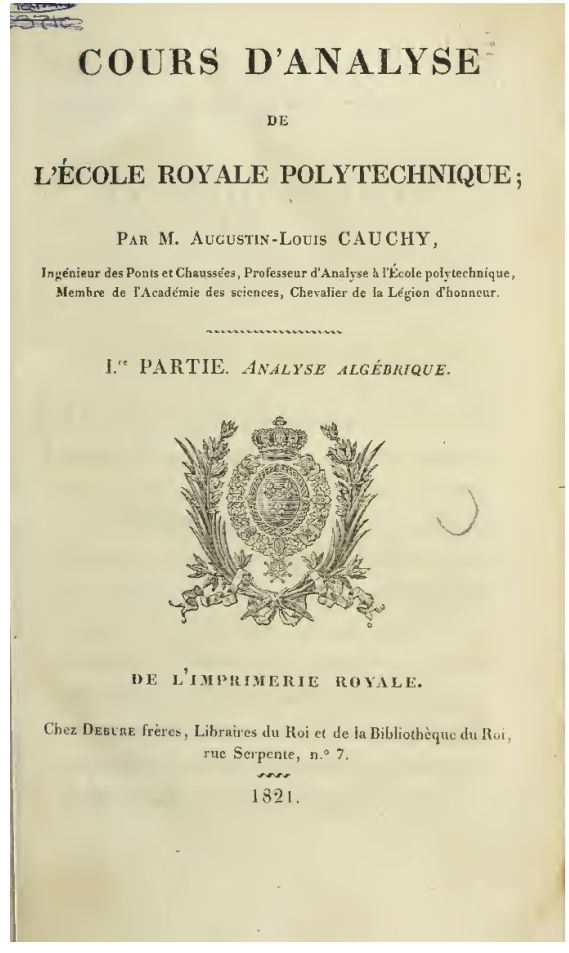
> *If a series of quantities has the property that the difference between its n-th term and every later one remains smaller than any given quantity . . . then there is always a certain constant quantity . . . which the terms of this series approach.*

Proof: The hypothesis that there exists a quantity X which the terms of this series approach . . . contains nothing impossible . . .

(See: *Mathematics emerging*, §16.1.1; for a full translation, see: S. B. Russ, A translation of Bolzano's paper on the intermediate value theorem, *Historia Mathematica* **7**(2) (1980), 156–185)

Cauchy's *Cours d'analyse*

Augustin-Louis Cauchy, *Cours d'analyse de l'École royale polytechnique* (1821)



COURS D'ANALYSE

DE

L'ÉCOLE ROYALE POLYTECHNIQUE;

PAR M. AUGUSTIN-LOUIS CAUCHY,

Ingénieur des Ponts et Chaussées, Professeur d'Analyse à l'École polytechnique, Membre de l'Académie des sciences, Chevalier de la Légion d'honneur.

I.re PARTIE. *ANALYSE ALGÉBRIQUE.*

DE L'IMPRIMERIE ROYALE.

Chez DEBURE frères, Libraires du Roi et de la Bibliothèque du Roi, rue Serpente, n.° 7.

1821.

(Annotated translation by Robert E. Bradley and C. Edward Sandifer, Springer, 2009)

Cauchy sequences: Cauchy (1821)

Augustin-Louis Cauchy, *Cours d'analyse* (1821), Ch. VI, pp. 124, 125:

> *In order for the series* $u_0,\ u_1,\ u_2, \ldots$ [that is, $\sum u_i$] *to be convergent ... it is necessary and sufficient that the partial sums*
>
> $$s_n = u_0 + u_1 + u_2 + \&c. \ldots + u_{n-1}$$
>
> *converge to a fixed limit* s*: in other words, it is necessary and sufficient that for infinitely large values of the number* n*, the sums*
>
> $$s_n,\ s_{n+1},\ s_{n+2},\ \&c. \ldots$$
>
> *differ from the limit* s*, and consequently from each other, by infinitely small quantities.*

(See: *Mathematics emerging*, §16.1.2.)

More from Cauchy (1821)

Further results from Cauchy's *Cours d'analyse*:

- ratio test;
- root test;
- alternating series test (proof uses Cauchy sequences);
- and many more.

(See *Mathematics emerging*, §16.1.2)

Cauchy sequences concluded

Early uses of Cauchy sequences:

- in da Cunha's *Principios mathematicos* (1782)
- in Bolzano's proof of the existence of a least upper bound (1817)
- in Cauchy's further results on sequences and series (1821)
- in Abel's proof of the general binomial theorem (1826)

BUT the convergence of Cauchy sequences themselves remained unproved

Continuity

Early definitions of continuity:

Wallis (1656): a curve that doesn't 'jump about'

Euler (1748): a curve described by a single expression

Later definitions of continuity:

Bolzano (1817): $f(x+\omega) - f(x)$ *can be made smaller than any given quantity, provided* ω *can be taken as small as we please*

Cauchy (1821): $f(x+a) - f(x)$ decreases with a

[Question: dependence? plagiarism? or a common source?]

Limits: early definitions

Wallis (1656): a quantity 'less than any assignable' quantity is zero

Newton (1687): adopted and 'proved' Wallis's definition; also used 'limit' in the sense of a 'bound' or 'ultimate value'; developed theory of 'first and last ratios'

D'Alembert (1751): 'one may approach a limit as closely as one wishes ... but never surpass it'; example: polygons and circle; he assumed that $\lim AB = \lim A \times \lim B$; a dictionary definition only — no theory

Limits: a later definition

Cauchy, *Cours d'analyse* (1821), p. 4:

> *When the values successively given to a variable approach indefinitely to a fixed value, so as to finish by differing from it by as little as one would wish, the latter is called the limit of all the others.*

Examples:

- an irrational number is a limit of rationals;
- in geometry a circle is a limit of polygons.

BUT still no formal definition of

- 'as small as one wishes',
- 'as closely as one wishes', ...

Differentiability: early ideas

For Leibniz and his immediate followers, any 'function' you could write down was automatically differentiable (by the usual rules).

For Lagrange, the 'Taylor' series

$$f(x+h) = f(x) + f'(x)h + \cdots$$

led naturally to consideration of

$$\frac{f(x+h) - f(x)}{h}$$

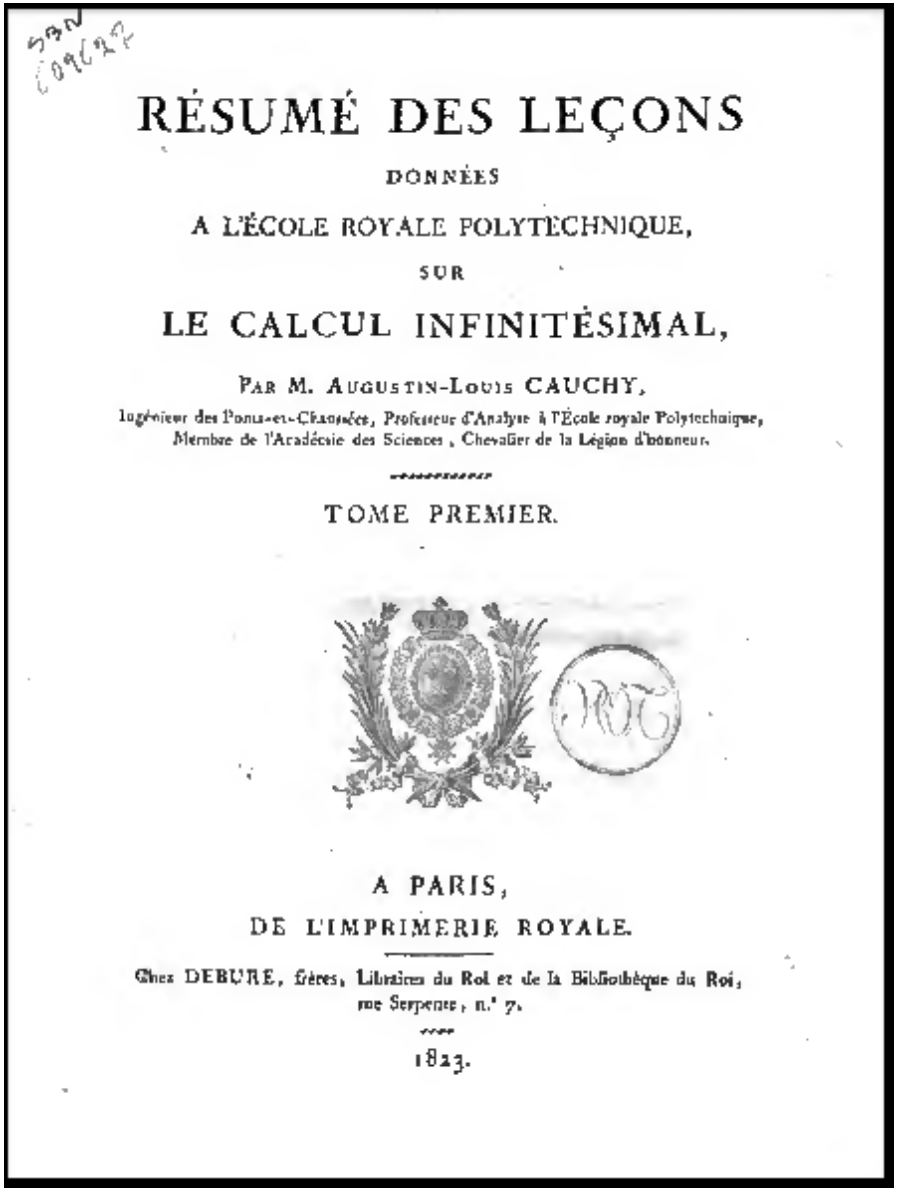
as an approximation to $f'(x)$, for small h

Ampère (1806) struggled with the meaning of

$$\frac{f(x+h) - f(x)}{h}$$

— why isn't it just zero or infinite?

Differentiability: Cauchy's *Résumé*



RÉSUMÉ DES LEÇONS

DONNÉES

A L'ÉCOLE ROYALE POLYTECHNIQUE,

SUR

LE CALCUL INFINITÉSIMAL,

PAR M. AUGUSTIN-LOUIS CAUCHY,

Ingénieur des Ponts-et-Chaussées, Professeur d'Analyse à l'École royale Polytechnique, Membre de l'Académie des Sciences, Chevalier de la Légion d'honneur.

TOME PREMIER.

A PARIS,

DE L'IMPRIMERIE ROYALE.

Chez DEBURE, frères, Libraires du Roi et de la Bibliothèque du Roi, rue Serpente, n.° 7.

1823.

Cauchy, *Résumé des leçons données à l'École royale polytechnique sur le calcul infinitésimal*, 1823

(Translation by Dennis M. Cates, Fairview Academic Press, 2012)

Differentiability: Cauchy's *Résumé*

> *. . . those who read my book will I hope be convinced that the principles of the differential calculus and its most important applications can easily be set out without the use of series.*

Defined the derivative as the limit of

$$\frac{f(x+h)-f(x)}{h}$$

with many particular examples: ax, a/x, $\sin x$, $\log x$, ...

but no concerns about existence in general

(See: *Mathematics emerging*, §14.1.4.)

Arbitrarily small intervals

A theorem of Lagrange (1797):

> *If the first derived function of a function f is strictly positive on an interval $[a, b]$, then $f(b) > f(a)$.*

Proof: Divide the interval $[a, b]$ into n subintervals, taking n as large as necessary ...

Unconvincing to modern eyes, but a useful technique.

(See: *Mathematics emerging*, §11.2.3.)

IVT revisited

Cauchy, *Cours d'analyse* (1821), Note III, p. 460 (On the numerical solution of equations):

Theorem: Let f be a real function of the variable x, which remains continuous with respect to this variable between the limits $x = x_0$, $x = X$. If the two quantities $f(x_0)$, $f(X)$ are of opposite signs, the equation $f(x) = 0$ will be satisfied by one or more real values of x contained between x_0 and X.

(See: *Mathematics emerging*, §11.2.6.)

IVT revisited

Cauchy's proof:

Choose $m > 1$. Divide the interval $[x_0, X]$ into m equal parts; find neighbouring division points x_1, X' such that $f(x_1)$, $f(X')$ are of opposite signs. Subdivide the interval $[x_1, X']$ into m equal parts; find neighbouring division points x_2, X'' such that $f(x_2)$, $f(X'')$ are of opposite signs. Continue in this way to obtain an increasing sequence $x_0,\ x_1, \ldots$ and a decreasing sequence $X,\ X', \ldots$. The difference $X^{(n)} - x_n$ is $(X - x_0)/m^n$, which may be made as small as one wishes. The sequences $x_0,\ x_1, \ldots$ and $X,\ X', \ldots$ therefore converge to a common limit a, at which $f(a) = 0$.

Note: Cauchy offered this as a fast method of approximation to roots of equations.

But it also provides a much more convincing proof of the Intermediate Value Theorem than that appearing earlier in Cauchy's text (*Cours d'analyse*, Ch. II, Theorem 4: p. 44).

ε and δ appear

A theorem of Cauchy, *Résumé* (1823):

Suppose that in the interval $[x_0, X]$ we have $A < f'(x) < B$. Then we also have

$$A < \frac{f(X) - f(x_0)}{X - x_0} < B$$

Proof: Choose two quantities $\epsilon,\ \delta, \ldots$ such that for $i < \delta$

$$f'(x) - \epsilon < \frac{f(x+i) - f(x)}{i} < f'(x) + \epsilon$$

etc.

(See: *Mathematics emerging*, §14.1.5.)

Hints of a broader class of functions

If a Taylor series exists for a given function, and all the coefficients vanish, then surely the function itself must vanish ...

However, Cauchy gave the example $f(x) = e^{-x^2} + e^{-x^{-2}}$, which is clearly never zero, but all of its derivatives vanish

So not every function can be expanded into a Taylor series, and it appears to be possible to conceive of functions to which the calculus is not immediately or naturally applicable ...

Modern rigour in analysis

Karl Weierstrass (1815–1897):

- taught at University of Berlin from 1856 onwards
- completed the rigorisation of calculus via systematic use of ε/δ methods

BUT we have no direct sources, only lecture notes or books by his pupils and followers

From France to Germany

By the later 19th century the mathematical centre of gravity in Europe had moved from the Parisian Écoles to the German universities:

Göttingen (est. 1734):	Gauss, Dirichlet, [Dedekind], Riemann, Klein, Hilbert, ...
Berlin (est. 1810):	Crelle (editor), Dirichlet, Eisenstein, Kummer, [Jacobi], Kronecker, Weierstrass, ...

with a focus on both research and teaching.

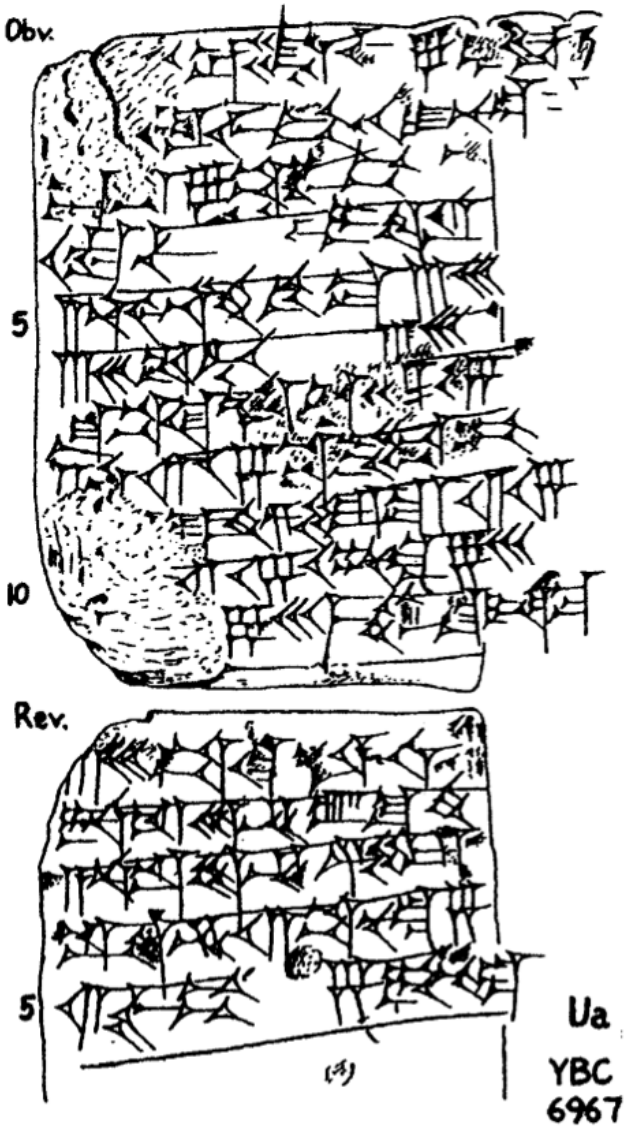
BO1 History of Mathematics
Lecture IX
Classical algebra: equation solving
1800 BC – AD 1800

MT 2022 Week 5

Summary

- Early quadratic equations
- Cubic and quartic equations
- Further 16th-century developments
- 17th century ideas
- 18th century ideas
- Looking back

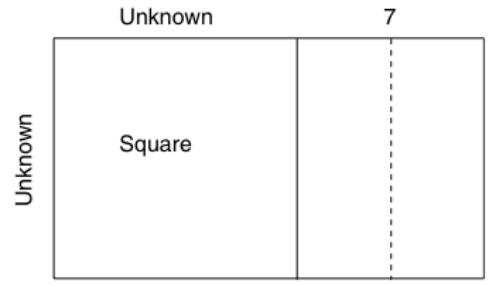
Completing the square, c. 1800 BC

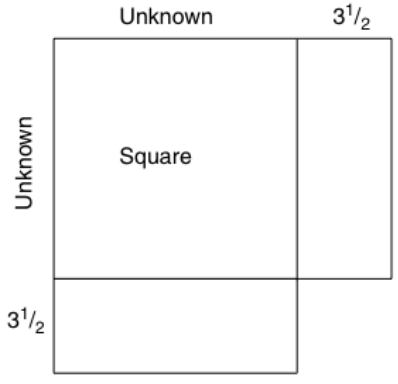


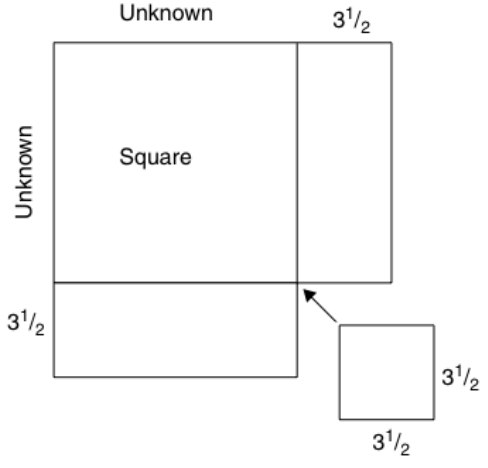
A Babylonian scribe, clay tablet BM 13901, c. 1800 BC:

A reciprocal exceeds its reciprocal by 7. What are the reciprocal and its reciprocal? Break in half the 7 by which the reciprocal exceeds its reciprocal, and $3\frac{1}{2}$ will come up. Multiply $3\frac{1}{2}$ by $3\frac{1}{2}$ and $12\frac{1}{4}$ will come up. Append 60, the area, to the $12\frac{1}{4}$ which came up for you and $72\frac{1}{4}$ will come up. What is the square-side of $72\frac{1}{4}$? $8\frac{1}{2}$. Put down $8\frac{1}{2}$ and $8\frac{1}{2}$ and subtract $3\frac{1}{2}$ from one of them; append $3\frac{1}{2}$ to one of them. One is 12, the other is 5. The reciprocal is 12, its reciprocal 5.

Completing the square, c. 1800 BC







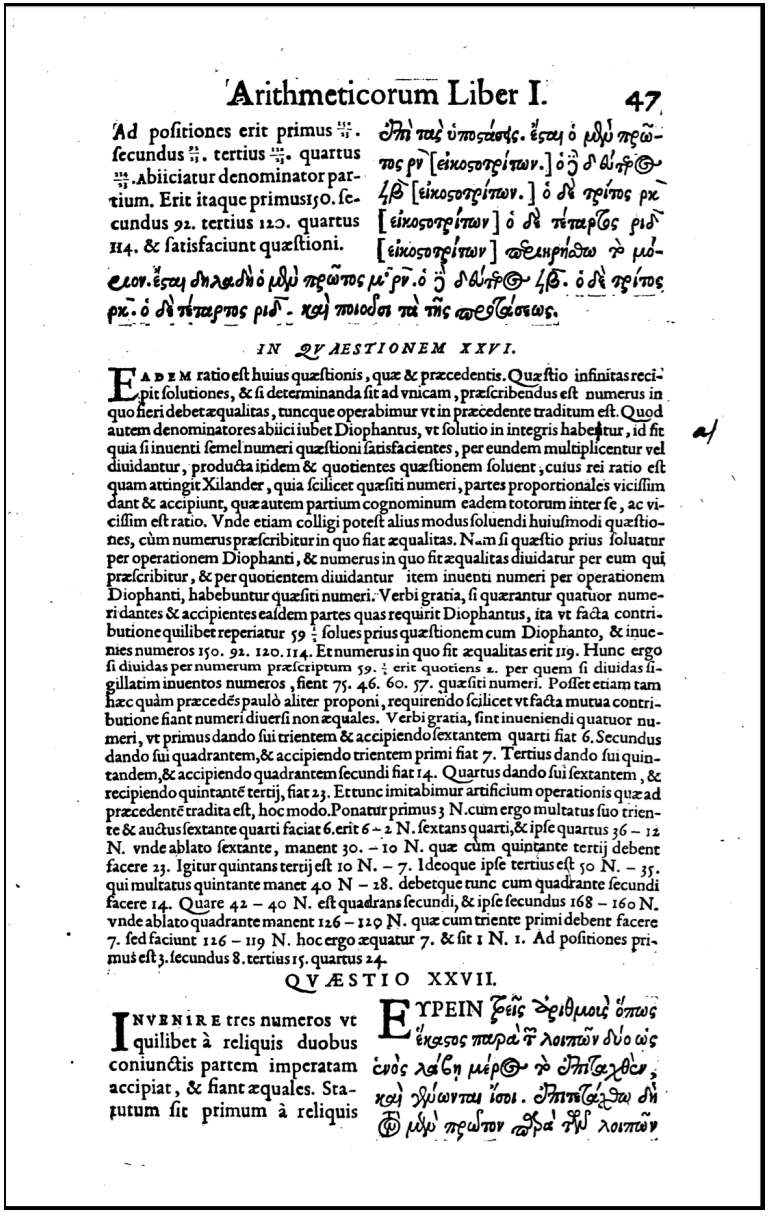
A Babylonian scribe, clay tablet BM 13901, c. 1800 BC:

A reciprocal exceeds its reciprocal by 7. What are the reciprocal and its reciprocal? Break in half the 7 by which the reciprocal exceeds its reciprocal, and $3\frac{1}{2}$ will come up. Multiply $3\frac{1}{2}$ by $3\frac{1}{2}$ and $12\frac{1}{4}$ will come up. Append 60, the area, to the $12\frac{1}{4}$ which came up for you and $72\frac{1}{4}$ will come up. What is the square-side of $72\frac{1}{4}$? $8\frac{1}{2}$. Put down $8\frac{1}{2}$ and $8\frac{1}{2}$ and subtract $3\frac{1}{2}$ from one of them; append $3\frac{1}{2}$ to one of them. One is 12, the other is 5. The reciprocal is 12, its reciprocal 5.

Points to note

- We have used the word ‘equation’ without writing down anything in symbols
- Solution recipe derived from geometrical insight
- Not (explicitly) a general solution — but reader ought to be able to adapt the method
- Is this algebra? Geometrical algebra?

Diophantus of Alexandria (3rd century AD)



Arithmeticorum Liber I. 47

Ad positiones erit primus $\frac{75}{114}$. secundus $\frac{39}{114}$. tertius $\frac{25}{114}$. quartus $\frac{15}{114}$. abiiciatur denominator partium. Erit itaque primus 150. secundus 92. tertius 120. quartus 114. & satisfaciunt quæstioni.

ἐπὶ τὰς ὑποστάσεις. ἔσται ὁ μὲν πρῶτος ρν [εἰκοσοστρίτων.] ὁ δὲ δεύτερος ξβ [εἰκοσοστρίτων.] ὁ δὲ τρίτος ρκ [εἰκοσοστρίτων] ὁ δὲ τέταρτος ριδ [εἰκοσοστρίτων] ᾧ συμπίπτει τὸ μόριον. ἔσται δηλαδὴ ὁ μὲν πρῶτος μ̄ρν. ὁ δὲ δεύτερος ξβ. ὁ δὲ τρίτος ρκ. ὁ δὲ τέταρτος ριδ. καὶ ποιοῦσι τὰ τῆς προτάσεως.

IN QVAESTIONEM XXVI.

EADEM ratio est huius quæstionis, quæ & præcedentis. Quæstio infinitas recipit solutiones, & si determinanda sit ad vnicam, præscribendus est numerus in quo fieri debet æqualitas, tuncque operabimur vt in præcedente traditum est. Quod autem denominatores abiici iubet Diophantus, vt solutio in integris habeatur, id fit quia si inuenti semel numeri quæstioni satisfacientes, per eundem multiplicentur vel diuidantur, producta itidem & quotientes quæstionem soluent, cuius rei ratio est quam attingit Xilander, quia scilicet quæsiti numeri, partes proportionales vicissim dant & accipiunt, quæ autem partium cognominum eadem totorum inter se, ac vicissim est ratio. Vnde etiam colligi potest alius modus soluendi huiusmodi quæstiones, cùm numerus præscribitur in quo fiat æqualitas. Nam si quæstio prius soluatur per operationem Diophanti, & numerus in quo fit æqualitas diuidatur per eum qui præscribitur, & per quotientem diuidantur item inuenti numeri per operationem Diophanti, habebuntur quæsiti numeri. Verbi gratia, si quærantur quatuor numeri dantes & accipientes easdem partes quas requirit Diophantus, ita vt facta contributione quilibet reperiatur 59 $\frac{1}{2}$: solues prius quæstionem cum Diophanto, & inuenies numeros 150. 92. 120. 114. Et numerus in quo fit æqualitas erit 119. Hunc ergo si diuidas per numerum præscriptum 59. $\frac{1}{2}$ erit quotiens 2. per quem si diuidas figillatim inuentos numeros, fient 75. 46. 60. 57. quæsiti numeri. Posset etiam tam hæc quàm præcedēs paulò aliter proponi, requirendo scilicet vt facta mutua contributione fiant numeri diuersi non æquales. Verbi gratia, sint inueniendi quatuor numeri, vt primus dando sui trientem & accipiendo sextantem quarti fiat 6. Secundus dando sui quadrantem, & accipiendo trientem primi fiat 7. Tertius dando sui quintandem, & accipiendo quadrantem secundi fiat 14. Quartus dando sui sextantem, & recipiendo quintantē tertij, fiat 23. Et tunc imitabimur artificium operationis quæ ad præcedentē tradita est, hoc modo. Ponatur primus 3 N. cum ergo multatus suo triente & auctus sextante quarti faciat 6. erit 6 – 2 N. sextans quarti, & ipse quartus 36 – 12 N. vnde ablato sextante, manent 30. – 10 N. quæ cùm quintante tertij debent facere 23. Igitur quintans tertij est 10 N. – 7. Ideoque ipse tertius est 50 N. – 35. qui multatus quintante manet 40 N – 28. debetque tunc cum quadrante secundi facere 14. Quare 42 – 40 N. est quadrans secundi, & ipse secundus 168 – 160 N. vnde ablato quadrante manent 126 – 120 N. quæ cum triente primi debent facere 7. sed faciunt 126 – 119 N. hoc ergo æquatur 7. & fit 1 N. 1. Ad positiones primus est 3. secundus 8. tertius 15. quartus 24.

QVÆSTIO XXVII.

INVENIRE tres numeros vt quilibet à reliquis duobus coniunctis partem imperatam accipiat, & fiant æquales. Statutum sit primum à reliquis

ΕΥΡΕΙΝ τρεῖς ἀριθμοὺς ὅπως ἕκαστος παρὰ τῶν λοιπῶν δύο ὡς ἐνὸς λαβὴ μέρος τὸ ὑποτεταγμένον, καὶ γίνωνται ἴσοι. ὑποκείσθω δὴ ὁ μὲν πρῶτος παρὰ τοὺς λοιποὺς

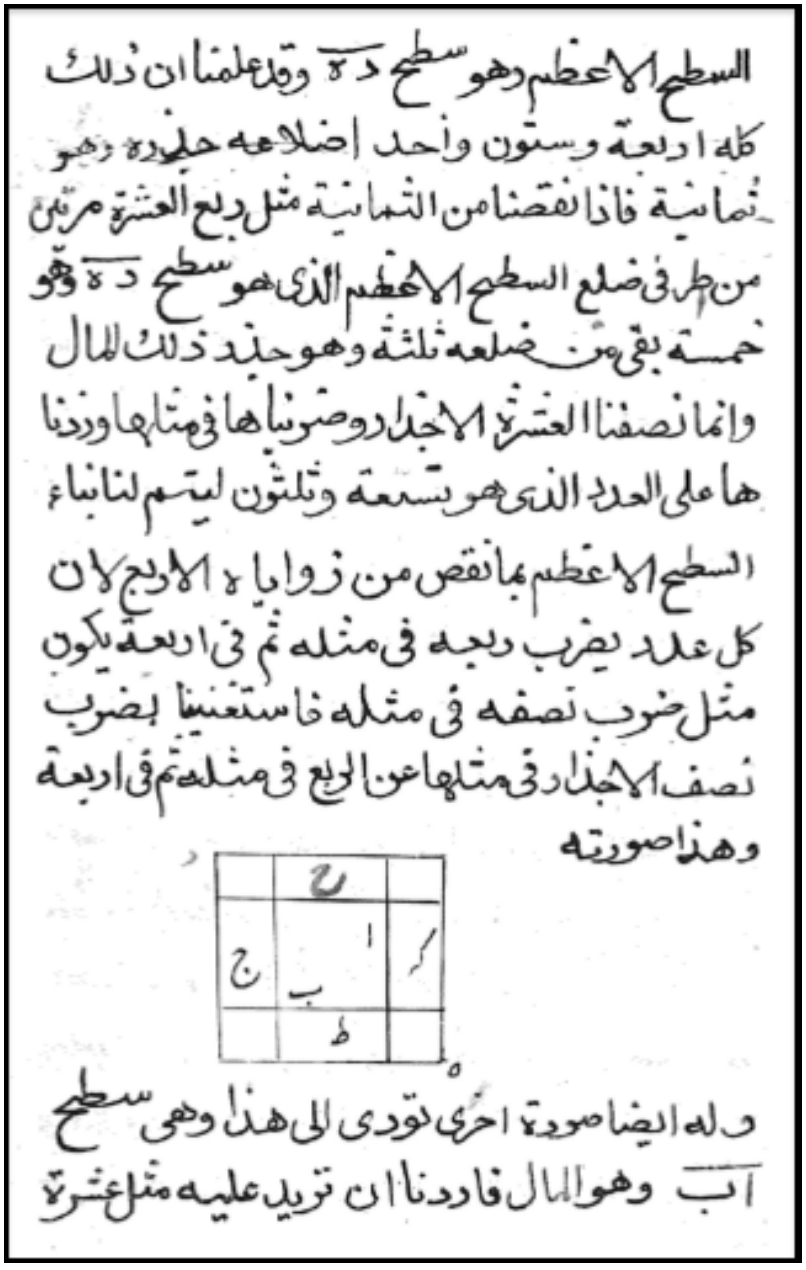
Problem I.27: Find two numbers such that their sum and product are given numbers

Muḥammad ibn Mūsā al-Khwārizmī (c. 780–c. 850)

Noted six cases of equations:

1. Squares are equal to roots ($ax^2 = bx$)
2. Squares are equal to numbers ($ax^2 = c$)
3. Roots are equal to numbers ($bx = c$)
4. Squares and roots are equal to numbers ($ax^2 + bx = c$)
5. Squares and numbers are equal to roots ($ax^2 + c = bx$)
6. Roots and numbers are equal to squares ($bx + c = ax^2$)

Muḥammad ibn Mūsā al-Khwārizmī (c. 780–c. 850)

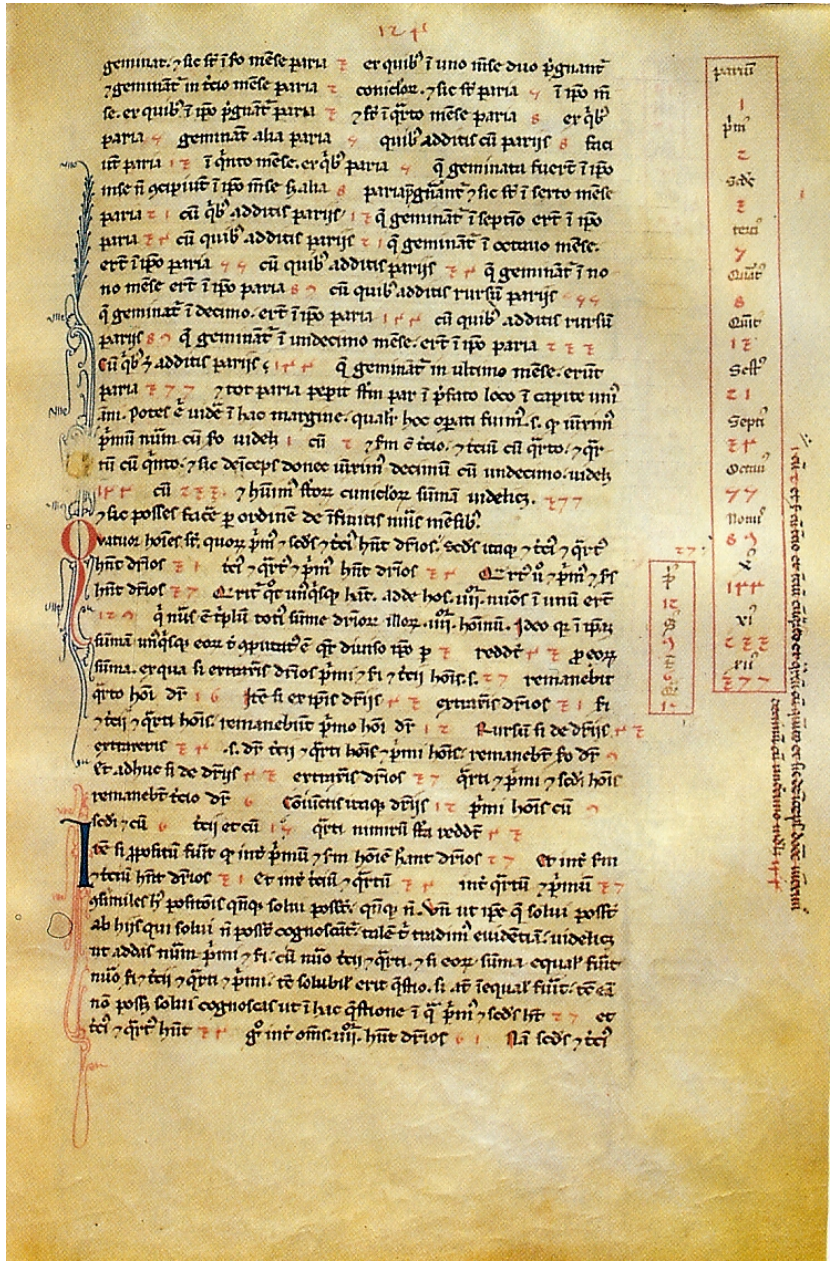

السطح الاعظم وهو سطح د ه وقد علمنا ان ذلك
كله اربعة وستون واحد اضلاعه جذر ده وهو
ثمانية فاذا نقصنا من الثمانية مثل ربع العشرة مرتين
من طرفي ضلع السطح الاعظم الذي هو سطح د ه وهو
خمسة بقي من ضلعه ثلثة وهو جذ ذلك المال
وانما نصفنا العشرة الاجذار وضربناها في مثلها وردنا
ها على العدد الذي هو تسعة وثلثون ليتم لنا بناء
السطح الاعظم بما نقص من زوايا ه الاربع لان
كل عدد يضرب ربعه في مثله ثم في اربعة يكون
مثل ضرب نصفه في مثله فاستغنينا بضرب
نصف الاجذار في مثلها عن الربع في مثله ثم في اربعة
وهذا صورته
ح
ا
ك
ج
ب
ط
وله ايضا صورة اخرى تؤدي الى هذا وهي سطح
آب وهو المال فاردنا ان نزيد عليه مثل عشرة

An algorithm for case (4) on the previous slide

Leonardo of Pisa (Fibonacci) (c. 1175–c. 1240/50)

Liber abaci (or *Liber abbaci*), Pisa, 1202:

- included al-Khwārizmi's recipes
- geometrical demonstrations and lots of examples
- didn't go very far beyond predecessors, but began transmission of Islamic ideas to Europe



Cubic equations (1)

Italy, early 16th century:

solutions to cubics of the form $x^3 + px = q$

- found by Scipione del Ferreo (or Ferro) (c. 1520)
- taught to Antonio Maria Fiore (pupil)
- and Annibale della Nave (son-in-law)
- rediscovered by Niccolò Tartaglia (1535)
- passed in rhyme to Girolamo Cardano (1539)

Cubic equations (2)

$$x^3 + px = q$$

When the cube with the things next after
Together equal some number apart
Find two others that by this differ
And this you will keep as a rule
That their product will always be equal
To a third cubed of the number of things
The difference then in general between
The sides of the cubes subtracted well
Will be your principal thing.

(Tartaglia, 1546; see: *Mathematics emerging*, §12.1.1)

Cubic equations (3)

$$x^3 + px = q$$

When the cube with the things next after
Together equal some number apart
Find two others that by this differ
And this you will keep as a rule
That their product will always be equal
To a third cubed of the number of things
The difference then in general between
The sides of the cubes subtracted well
Will be your principal thing.

Interpretation of Tartaglia's rhyme:

Find u, v such that

$$u - v = q, \quad uv = \left(\frac{p}{3}\right)^3.$$

Then

$$x = \sqrt[3]{u} - \sqrt[3]{v}$$

NB: In an equation $y^3 + ay^2 + by + c = 0$ we can put $y = x - \frac{a}{3}$ to remove the square term, so this solution is general.

Cubic equations (4)

In modern terms, one of the solutions of the equation $ax^3 + bx^2 + cx + d = 0$ has the form

$$x = \sqrt[3]{\left(-\frac{b^3}{27a^3} + \frac{bc}{6a^2} - \frac{d}{2a}\right) + \sqrt{\left(-\frac{b^3}{27a^3} + \frac{bc}{6a^2} - \frac{d}{2a}\right)^2 + \left(\frac{c}{3a} - \frac{b^2}{9a^2}\right)^3}} + \sqrt[3]{\left(-\frac{b^3}{27a^3} + \frac{bc}{6a^2} - \frac{d}{2a}\right) - \sqrt{\left(-\frac{b^3}{27a^3} + \frac{bc}{6a^2} - \frac{d}{2a}\right)^2 + \left(\frac{c}{3a} - \frac{b^2}{9a^2}\right)^3}} - \frac{b}{3a}$$

with similar expressions (in radicals) for the remaining two roots

Cardano's *Ars magna, sive de regulis algebraicis* (1545)

HIERONYMI CAR
DANI, PRÆSTANTISSIMI MATHE-
MATICI, PHILOSOPHI, AC MEDICI,
ARTIS MAGNÆ,
SIVE DE REGVLIS ALGEBRAICIS,
Lib. unus. Qui & totius operis de Arithmetica, quod
OPVS PERFECTVM
inscripsit, est in ordine Decimus.

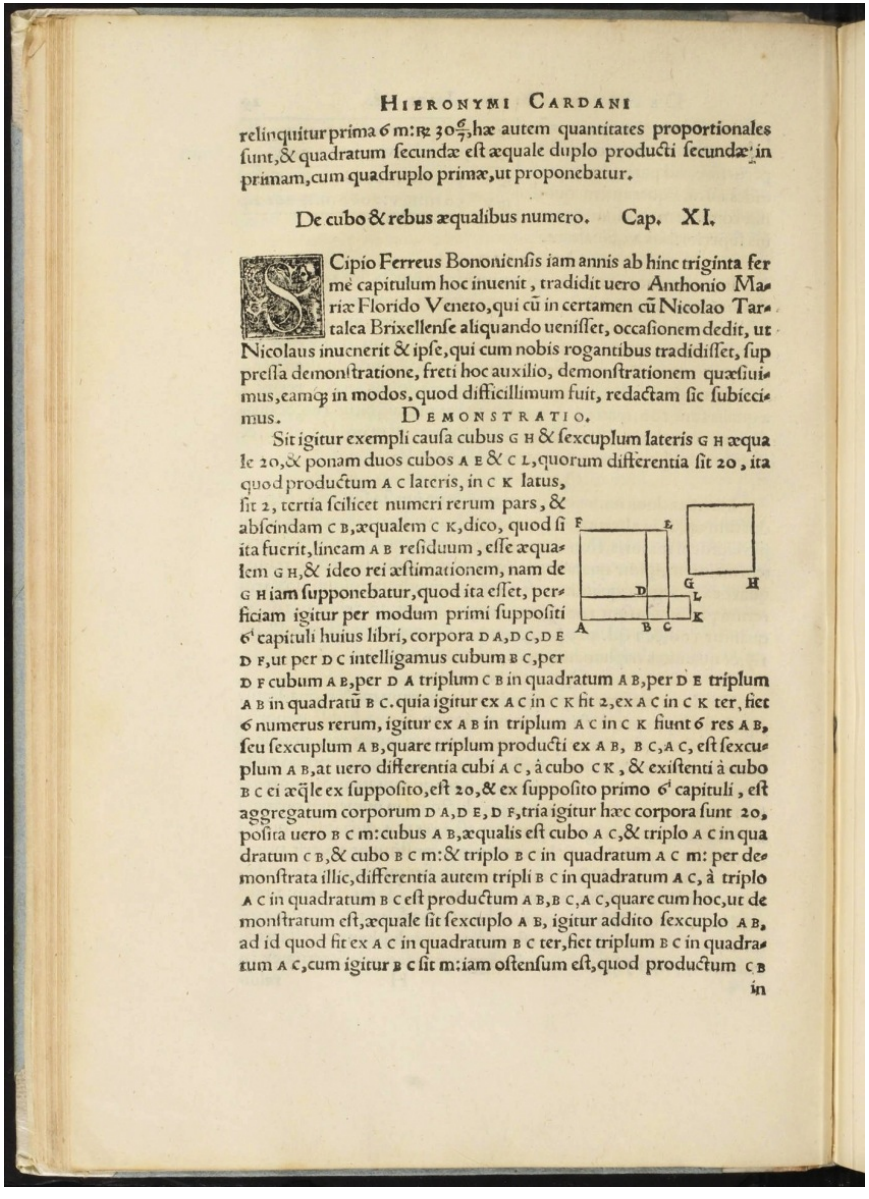
HAbes in hoc libro, studiose Lector, Regulas Algebraicas (Itali, de la Cosa uocant) nouis adinuentionibus, ac demonstrationibus ab Authore ita locupletatas, ut pro pauculis antea uulgò tritis, iam septuaginta euaserint. Neq; solum, ubi unus numerus alteri, aut duo uni, uerum etiam, ubi duo duobus, aut tres uni æquales fuerint, nodum explicant. Hunc aũt librum ideo seorsim edere placuit, ut hoc abstrusissimo, & planè inexhausto totius Arithmeticæ thesauro in lucem eruto, & quasi in theatro quodam omnibus ad spectandum exposito, Lectores incitarẽtur, ut reliquos Operis Perfecti libros, qui per Tomos edentur, tanto auidius amplectantur, ac minore fastidio perdiscant.

3 Numerus & positio, & ignota quantitas.
4 Numerus & q̃dratũ positionis, & ignota quantitas, seu numerus & q̃d^tum quantitatis ignotæ & positio, seu numerus cũ q̃d^to positionis quantitatis ignotæ, seu numerus & productum ex positione in quantitatem ignotam, cum altera earum, uel cum q̃drato unius earum.

Capitula primitiua.	Capitula deriuatiua.
1 Numerus æqualis q̃d^to & rebus.	1 Numerus æq̃lis q̃d' q̃d' & q̃d'.
	2 Numerus æqualis cu' q̃d' & cub'.
2 Numerus & res æqualia quadrato.	3 Numerus & q̃d' æquales q̃d' q̃d'.
	4 Numerus & cub' æq̃les cub' q̃d'.
3 Numerus & q̃d' ęqualia rebus.	5 Numerus & q̃d' q̃d' æqualia q̃d'.
	6 Numerus & cub' q̃d' æq̃lia cub'.
4 Numerus æqualis cubo & rebus.	7 Numerus æq̃lis q̃d' & cub' q̃d'.
	8 Numerus æq̃lis cub' & cubo cubi.
5 Numerus & res æqualia cubis.	9 Numerus & q̃d' æq̃lia cub' q̃d'.
	10 Numerus & cub' æq̃lia cub' cub'.
6 Numerus & cubus æq̃l' rebus æq̃tio prima.	11 Numerus & cub' q̃d' ęq̃l' q̃d' ęq̃° pri^a.
	12 Nu^r' & cub' cub' æq̃lia cub' ęq̃° pri^a.
7 Numerus & cub' ęq̃lia rebus ęq̃° secunda.	13 Nu^r' & cub' q̃d' ęq̃lia q̃d' ęq̃° secunda.
	14 Nu^r' & cub' cub' ęq̃lia cub' ęq̃° secũd^a.
8 Numerus ęq̃lis q̃drato & cubo.	15 Numerus ęq̃lis q̃d' q̃d' & cub' q̃drati.
	16 Numerus ęq̃lis cub' q̃d' & cub' cubi.
9 Numerus & q̃dratum æqualia cubo.	17 Numerus & q̃d' q̃d' ęq̃lia cub' q̃drati.
	18 Numerus & cub' q̃d' ęq̃lia cub' cubi.
10 Numerus & cub' ęq̃lia q̃drato ęq̃° prima.	19 Nu^r' & cub' q̃d' ęq̃lia q̃d' q̃d' æq̃° pri^a.
	20 Nu^r' & cub' cub' ęq̃lia cub' q̃d' ęq̃° pri^a.
11 Numerus & cub' ęq̃lia q̃drato ęq̃° secunda.	21 Nu^r' & cub' q̃d' ęq̃l' q̃d' q̃d' ęq̃° secũd^a.
	22 Nu^r' & cub' cub' ęq̃l' cu' q̃d' ęq̃° secũd.
12 Numerus ęq̃lis rebus q̃drato & cubo.	23 Nu^r' ęq̃lis q̃d' & q̃d' q̃d' & cub' q̃d'.
	24 Nu^r' ęq̃lis cub' & cub' q̃d' & cub' cu'.
13 Numerus & res ęq̃lia quadrato & cubo.	25 Nu^r' & q̃d' ęq̃lia q̃d' q̃d' & cub' q̃d'.
	26 Nu^r' & cub' ęq̃lia cub' q̃d & cub' cu'.
14 Numerus & res & q̃d' ęqualia cubo.	27 Nu^r' & q̃d' & q̃d' q̃d' ęq̃lia cub' q̃d'.
	28 Nu^r' & cub' & cub q̃d ęq̃lia cu cub'.
15 Numerus & q̃d' ęq̃lia rebus & cub' ęq̃° prima.	29 Nu^r' & q̃d' q̃d' ęq̃l' q̃d' & cu' q̃d ęq̃° p^a
	30 Nu^r' & cu' q̃d' ęq̃l' cu & cu' cu ęq̃° pri^a.
16 Numerus & q̃d' ęq̃lia rebus & cubo ęq̃° secũda.	31 Nu^r' & q̃d' q̃d' ęq̃l' q̃d' & cu' q̃d ęq̃° sec.
	32 Nu^r' & cu' q̃d' ęq̃l' cu' & cu' cu' ęq̃° sec.

17 Nume-

Cardano on the cubic



HIERONYMI CARDANI

relinquitur prima 6 m: R: 30 6/7, hæ autem quantitates proportionales sunt, & quadratum secundæ est æquale duplo producti secundæ in primam, cum quadruplo primæ, ut proponebatur.

De cubo & rebus æqualibus numero. Cap. XI.

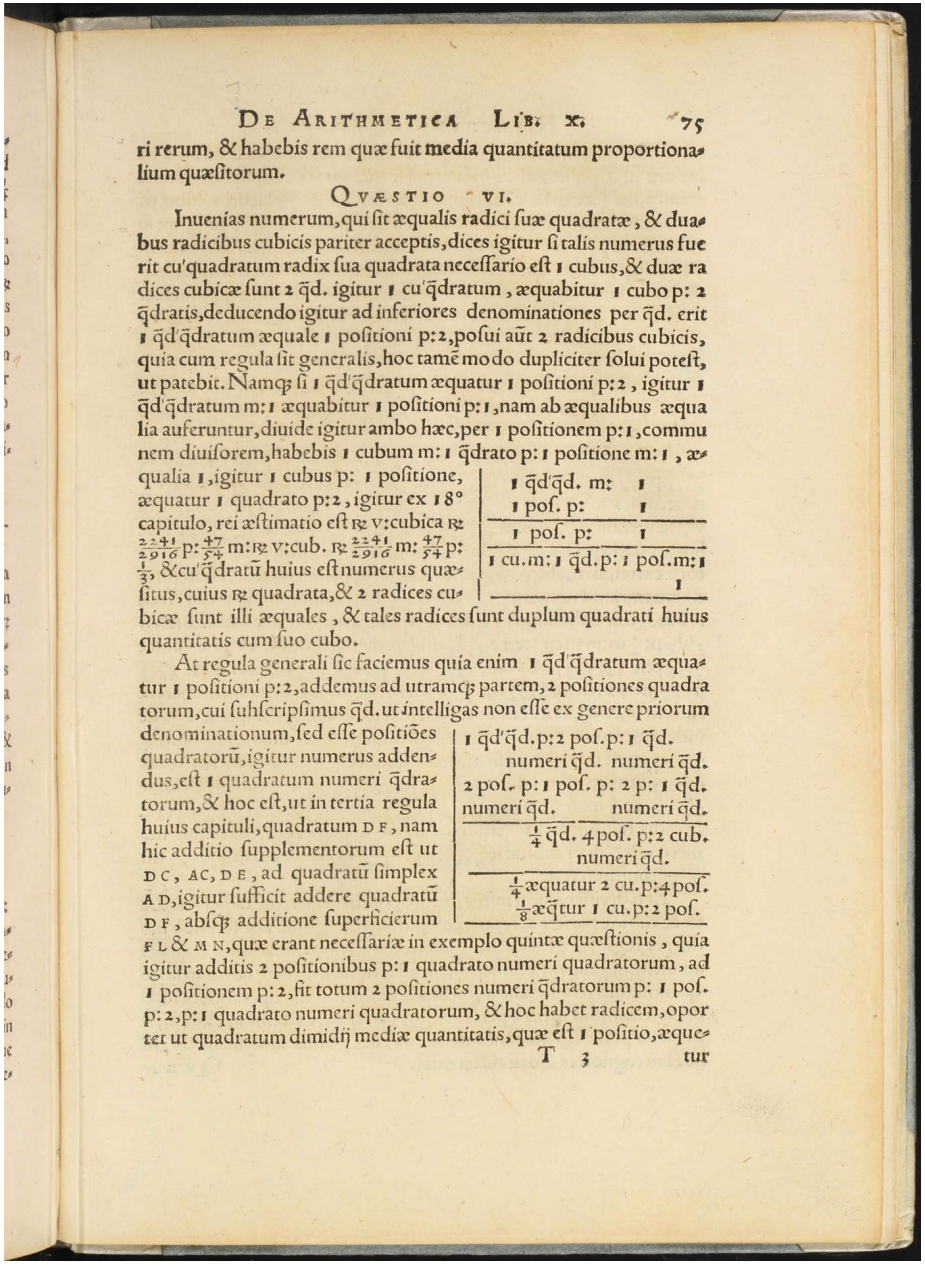
SCipio Ferreus Bononiensis iam annis ab hinc triginta fer mè capitulum hoc inuenit, tradidit uero Anthonio Ma riæ Florido Veneto, qui cũ in certamen cũ Nicolao Tar talea Brixellense aliquando uenisset, occasionem dedit, ut Nicolaus inuenerit & ipse, qui cum nobis rogantibus tradidisset, sup pressa demonstratione, freti hoc auxilio, demonstrationem quæsiui mus, eamq; in modos, quod difficillimum fuit, redactam sic subicci mus. DEMONSTRATIO.

Sit igitur exempli causa cubus G H & sexcuplum lateris G H æqua le 20, & ponam duos cubos A E & C L, quorum differentia sit 20, ita quod productum A C lateris, in C K latus, sit 2, tertia scilicet numeri rerum pars, & abscindam C B, æqualem C K, dico, quod si ita fuerit, lineam A B residuum, esse æqua lem G H, & ideo rei æstimationem, nam de G H iam supponebatur, quod ita esset, per ficiam igitur per modum primi suppositi 6^i capituli huius libri, corpora D A, D C, D E D F, ut per D C intelligamus cubum B C, per D F cubum A B, per D A triplum C B in quadratum A B, per D E triplum A B in quadratũ B C. quia igitur ex A C in C K fit 2, ex A C in C K ter, fiet 6 numerus rerum, igitur ex A B in triplum A C in C K fiunt 6 res A B, seu sexcuplum A B, quare triplum producti ex A B, B C, A C, est sexcuplum A B, at uero differentia cubi A C, à cubo C K, & existenti à cubo B C ei æq̃le ex supposito, est 20, & ex suppositio primo 6^i capituli, est aggregatum corporum D A, D E, D F, tria igitur hæc corpora sunt 20, posita uero B C m: cubus A B, æqualis est cubo A C, & triplo A C in qua dratum C B, & cubo B C m: & triplo B C in quadratum A C m: per de monstrata illic, differentia autem tripli B C in quadratum A C, à triplo A C in quadratum B C est productum A B, B C, A C, quare cum hoc, ut de monstratum est, æquale sit sexcuplo A B, igitur addito sexcuplo A B, ad id quod fit ex A C in quadratum B C ter, fiet triplum B C in quadra tum A C, cum igitur B C sit m: iam ostensum est, quod productum C B in

- Geometrical justification remains
- General solution (to particular case), rather than example to be followed
- Make substitution $x = y - \frac{a}{3}$ in $y^3 + ax^2 + bx + c = d$ to suppress square term and obtain equation of the form $x^3 + px = q$ — manipulation of equations prior to solution

Quartic equations (1)

General solution discovered (again on a case-by-case basis) by Lodovico Ferrari (c. 1540) and published by Cardano, in the form of worked examples, alongside solution of cubic



DE ARITHMETICA LIB. X. 75

ri rerum, & habebis rem quæ fuit media quantitatum proportionalium quæſitorum.

QVÆSTIO VI.

Inuenias numerum, qui ſit æqualis radici ſuæ quadratæ, & duabus radicibus cubicis pariter acceptis, dices igitur ſi talis numerus fuerit cu'quadratum radix ſua quadrata neceſſario eſt 1 cubus, & duæ radices cubicæ ſunt 2 q̃d. igitur 1 cu'q̃dratum, æquabitur 1 cubo p: 2 q̃dratis, deducendo igitur ad inferiores denominationes per q̃d. erit 1 q̃d'q̃dratum æquale 1 poſitioni p: 2, poſui aũt 2 radicibus cubicis, quia cum regula ſit generalis, hoc tamẽ modo dupliciter ſolui poteſt, ut patebit. Namq; ſi 1 q̃d'q̃dratum æquatur 1 poſitioni p: 2, igitur 1 q̃d'q̃dratum m: 1 æquabitur 1 poſitioni p: 1, nam ab æqualibus æqualia auferuntur, diuide igitur ambo hæc, per 1 poſitionem p: 1, commu nem diuiſorem, habebis 1 cubum m: 1 q̃drato p: 1 poſitione m: 1, æqualia 1, igitur 1 cubus p: 1 poſitione, æquatur 1 quadrato p: 2, igitur ex 18° capitulo, rei æſtimatio eſt ℞ v: cubica ℞ $\frac{2241}{2916}$ p: $\frac{47}{54}$ m: ℞ v: cub. ℞ $\frac{2241}{2916}$ m: $\frac{47}{54}$ p: $\frac{1}{3}$, & cu'q̃dratũ huius eſt numerus quæſitus, cuius ℞ quadrata, & 2 radices cubicæ ſunt illi æquales, & tales radices ſunt duplum quadrati huius quantitatis cum ſuo cubo.

1 q̃d'q̃d. m:	1
1 poſ. p:	1
1 poſ. p:	1
1 cu. m: 1 q̃d. p: 1 poſ. m:	1

At regula generali ſic faciemus quia enim 1 q̃d'q̃dratum æquatur 1 poſitioni p: 2, addemus ad utramq; partem, 2 poſitiones quadratorum, cui ſubſcripſimus q̃d. ut intelligas non eſſe ex genere priorum denominationum, ſed eſſe poſitiões quadratorũ, igitur numerus addendus, eſt 1 quadratum numeri q̃dratorum, & hoc eſt, ut in tertia regula huius capituli, quadratum D F, nam hic additio ſupplementorum eſt ut D C, A C, D E, ad quadratũ ſimplex A D, igitur ſufficit addere quadratũ D F, abſq; additione ſuperficierum F L & M N, quæ erant neceſſariæ in exemplo quintæ quæſtionis, quia igitur additis 2 poſitionibus p: 1 quadrato numeri quadratorum, ad 1 poſitionem p: 2, fit totum 2 poſitiones numeri q̃dratorum p: 1 poſ. p: 2, p: 1 quadrato numeri quadratorum, & hoc habet radicem, oportet ut quadratum dimidij mediæ quantitatis, quæ eſt 1 poſitio, æque-

1 q̃d'q̃d. p: 2 poſ. p: 1 q̃d.	
numeri q̃d.	numeri q̃d.
2 poſ. p: 1 poſ. p: 2 p: 1 q̃d.	
numeri q̃d.	numeri q̃d.
$\frac{1}{4}$ q̃d. 4 poſ. p: 2 cub.	
numeri q̃d.	
$\frac{1}{4}$ æquatur 2 cu. p: 4 poſ.	
$\frac{1}{8}$ æq̃tur 1 cu. p: 2 poſ.	

Quartic equations (2)

In modern terms, suppose that

$$x^4 = px^2 + qx + r.$$

Add $2yx^2 + y^2$ to each side to give

$$(x^2 + y)^2 = (p + 2y)x^2 + qx + (r + y^2).$$

Now we seek y such that the right hand side is a perfect square:

$$8y^3 + 4py^2 + 8ry + (4pr - q^2) = 0.$$

So the problem is reduced to solving a cubic equation and then a quadratic.

NB: In an equation $y^4 + ay^3 + by^2 + cy + d = 0$ we can put $y = x - \frac{a}{4}$ to remove the cube term, so this solution is general.

Quartic equations (3)

Formulae for the solutions of the general quartic equation, in all their unedifying glory, may be found at:

http://planetmath.org/QuarticFormula

Cardano's *Ars Magna* may also be found online here

Further 16th-century developments

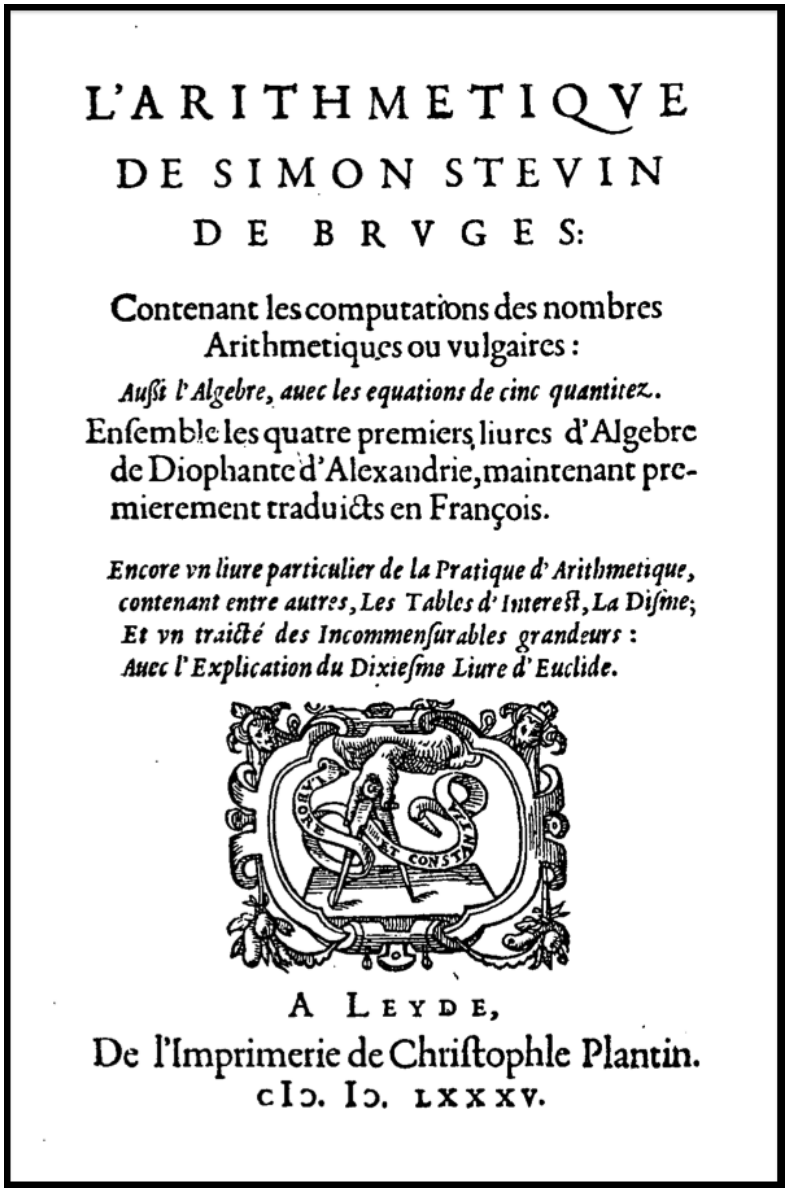
L'ALGEBRA
OPERA...
Di RAFAEL BOMBELLI da Bologna
...Diuisa in tre Libri.
Con la quale ciascuno da se potrà venire in perfetta cognitione della teorica dell'Arimetica.
Con vna Tauola copiosa delle materie, che in ella si contengono.
Posta hora in luce à beneficio delli Studiosi di detta professione.
IN BOLOGNA,
Per Giouanni Rossi. MDLXXIX.
Con licenza de' Superiori.

Rafael Bombelli, *L'algebra* (1572):

- heavily influenced by Cardano
- equation solving, new notation
- exploration of complex numbers

  [to be dealt with in a later lecture]

Further 16th-century developments



L'ARITHMETIQVE
DE SIMON STEVIN
DE BRVGES:

Contenant les computations des nombres Arithmetiques ou vulgaires :
Auſsi l'Algebre, auec les equations de cinc quantitez.
Enſemble les quatre premiers liures d'Algebre de Diophante d'Alexandrie, maintenant premierement traduicts en François.

Encore vn liure particulier de la Pratique d'Arithmetique, contenant entre autres, Les Tables d'Intereſt, La Diſme; Et vn traicté des Incommenſurables grandeurs : Auec l'Explication du Dixieſme Liure d'Euclide.

A LEYDE,
De l'Imprimerie de Chriſtophle Plantin.
cIↃ. IↃ. LXXXV.

Simon Stevin, *L'arithmetique ... aussi l'algebre* (1585):

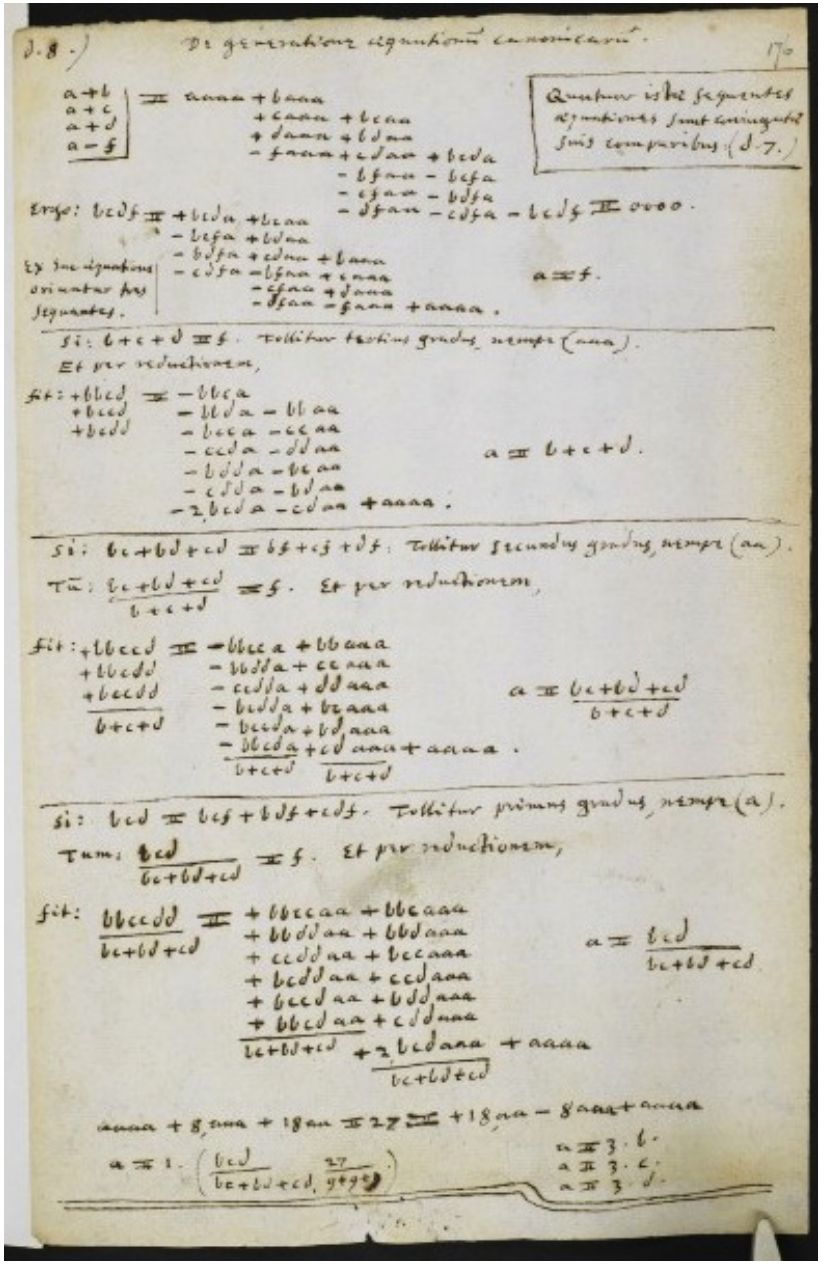
- heavily influenced by Cardano through Bombelli
- appended his treatise on decimal notation

Further 16th-century developments

François Viète (1590s):

- links between algebra and geometry
- (algebra as 'analysis' or 'analytic art')
- notation [recall Lecture III]
- numerical methods for solving equations

Thomas Harriot (c. 1600)

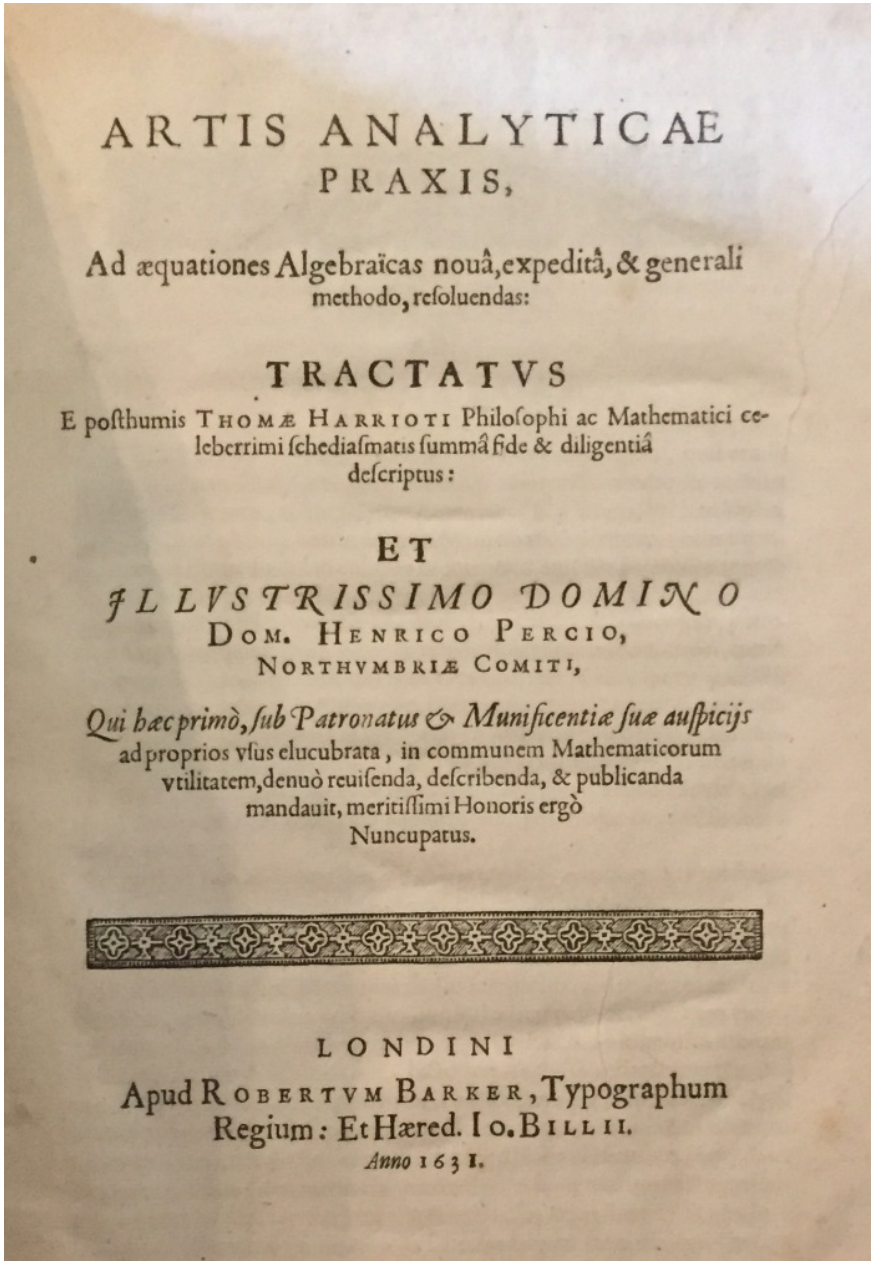


Add MS 6783 f. 176

Note:

- notation [see lecture III];
- appearance of polynomials as products of linear factors.

Thomas Harriot (1631)



ARTIS ANALYTICAE
PRAXIS,

Ad æquationes Algebraïcas nouâ, expeditâ, & generali methodo, reſoluendas:

TRACTATVS

E poſthumis THOMÆ HARRIOTI Philoſophi ac Mathematici celeberrimi ſchediaſmatis ſummâ fide & diligentiâ deſcriptus:

ET

ILLVSTRISSIMO DOMINO
DOM. HENRICO PERCIO,
NORTHVMBRIÆ COMITI,

Qui hæc primò, ſub Patronatus & Munificentiæ ſuæ auſpicijs ad proprios vſus elucubrata, in communem Mathematicorum vtilitatem, denuò reuiſenda, deſcribenda, & publicanda mandauit, meritiſſimi Honoris ergò Nuncupatus.

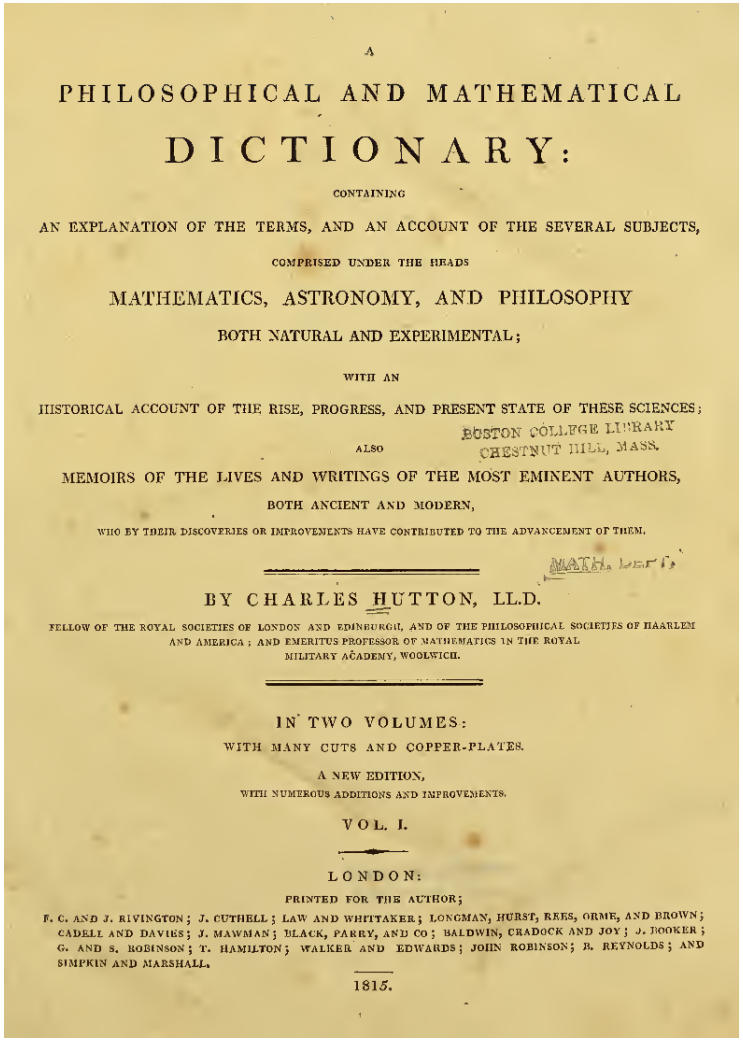
LONDINI
Apud ROBERTVM BARKER, Typographum Regium: Et Hæred. Io. BILLII.
Anno 1631.

Some of Harriot's ideas found their way into his *Artis analyticae praxis* (*The practice of the analytic art*), published posthumously in 1631

But editors did not permit negative or imaginary roots [to be discussed further in a later lecture]

See *Mathematics emerging*, §12.2.1.

Commentary on Harriot



A

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AN EXPLANATION OF THE TERMS, AND AN ACCOUNT OF THE SEVERAL SUBJECTS,

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BOTH ANCIENT AND MODERN,

WHO BY THEIR DISCOVERIES OR IMPROVEMENTS HAVE CONTRIBUTED TO THE ADVANCEMENT OF THEM.

MATH. DEPT.

BY CHARLES HUTTON, LL.D.

FELLOW OF THE ROYAL SOCIETIES OF LONDON AND EDINBURGH, AND OF THE PHILOSOPHICAL SOCIETIES OF HAARLEM AND AMERICA; AND EMERITUS PROFESSOR OF MATHEMATICS IN THE ROYAL MILITARY ACADEMY, WOOLWICH.

IN TWO VOLUMES:

WITH MANY CUTS AND COPPER-PLATES.

A NEW EDITION,

WITH NUMEROUS ADDITIONS AND IMPROVEMENTS.

VOL. I.

LONDON:

PRINTED FOR THE AUTHOR;

F. C. AND J. RIVINGTON; J. CUTHELL; LAW AND WHITAKER; LONGMAN, HURST, REES, ORME, AND BROWN; CADELL AND DAVIES; J. MAWMAN; BLACK, PARRY, AND CO; BALDWIN, CRADOCK AND JOY; J. BOOKER; G. AND S. ROBINSON; T. HAMILTON; WALKER AND EDWARDS; JOHN ROBINSON; B. REYNOLDS; AND SIMPKIN AND MARSHALL.

1815.

Charles Hutton, *A mathematical and philosophical dictionary*, London, 1795, vol. 1, p. 91 (p. 96 of revised edition, 1815):

He shewed the universal generation of all the compound or affected equations, by the continual multiplication of so many simple ones; thereby plainly exhibiting to the eye the whole circumstances of the nature, mystery and number of the roots of equations; with the composition and relations of the coefficients of the terms; . . .

Algebra in the 17th century

From 1600 onwards, 'algebra' as a set of recipes and techniques began to diverge in two (linked) directions:

- 'algebra' as a tool or a language (a.k.a. 'analysis' or the 'analytic art')
- 'algebra' as an object of study in its own right (the 'theory of equations')

Descartes on algebra

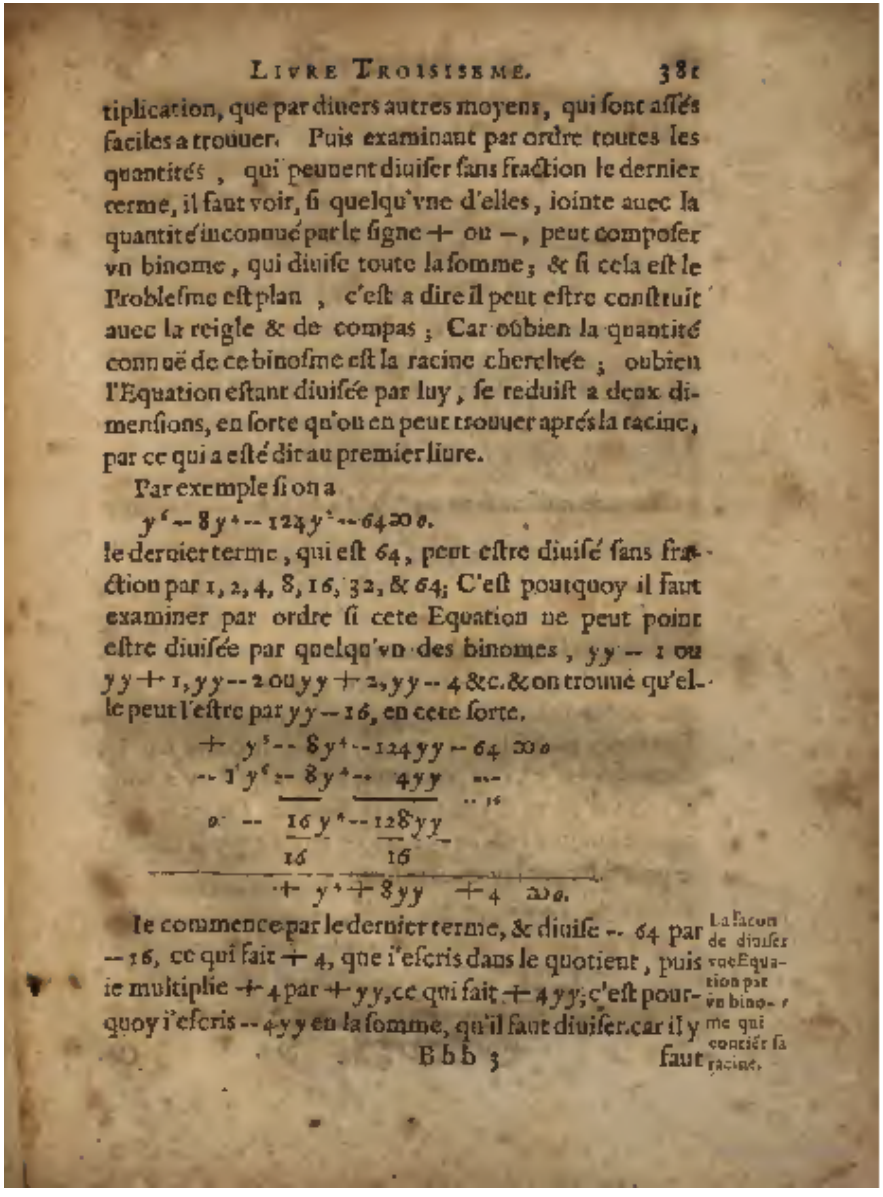
Polynomials feature in Descartes' *La géométrie* (1637), e.g.:

- one example to show that polynomials can be constructed from their roots (influenced by Harriot?);
- 'rule of signs': the number of positive ('true') roots of a polynomial is at most the number of times that the sign changes as we read term-by-term; the number of negative ('false') roots is at most the number of successions of the same sign; for example,

$$x^4 - 4x^3 - 19xx + 106x - 120 = 0$$

  has at most 3 positive roots and at most one negative;
- can always make a transformation to remove the second-highest term.

Descartes on cubics



LIVRE TROISIESME. 381

tiplication, que par diuers autres moyens, qui ſont aſſés faciles a trouuer. Puis examinant par ordre toutes les quantités, qui peuuent diuiſer ſans fraction le dernier terme, il faut voir, ſi quelqu'vne d'elles, iointe auec la quantité inconnuë par le ſigne + ou −, peut compoſer vn binome, qui diuiſe toute la ſomme; & ſi cela eſt le Probleſme eſt plan, c'eſt a dire il peut eſtre conſtruit auec la reigle & de compas; Car oûbien la quantité connuë de ce binoſme eſt la racine cherchée; oubien l'Equation eſtant diuiſée par luy, ſe reduiſt a deux dimenſions, en ſorte qu'on en peut trouuer aprés la racine, par ce qui a eſté dit au premier liure.

Par exemple ſi on a

$y^6 - 8y^4 - 124y^2 - 64 \propto 0.$

le dernier terme, qui eſt 64, peut eſtre diuiſé ſans fraction par 1, 2, 4, 8, 16, 32, & 64; C'eſt pourquoy il faut examiner par ordre ſi cete Equation ne peut point eſtre diuiſée par quelqu'vn des binomes, $yy - 1$ ou $yy + 1$, $yy - 2$ ou $yy + 2$, $yy - 4$ &c. & on trouue qu'elle peut l'eſtre par $yy - 16$, en cete ſorte.

$$\begin{array}{l} + y^6 - 8y^4 - 124yy - 64 \propto 0 \\ - y^6 - 8y^4 - 4yy \quad \text{\&c.} \\ \hline 0 - 16y^4 - 128yy \\ \quad\; 16 \qquad 16 \\ \hline + y^4 + 8yy + 4 \propto 0. \end{array}$$

Ie commence par le dernier terme, & diuiſe − 64 par − 16, ce qui fait + 4, que i'eſcris dans le quotient, puis ie multiplie + 4 par + yy, ce qui fait + 4yy; c'eſt pourquoy i'eſcris − 4yy en la ſomme, qu'il faut diuiſer. car il y faut

La façon de diuiſer vne Equation par vn binome qui contiēt ſa racine.

Bbb 3

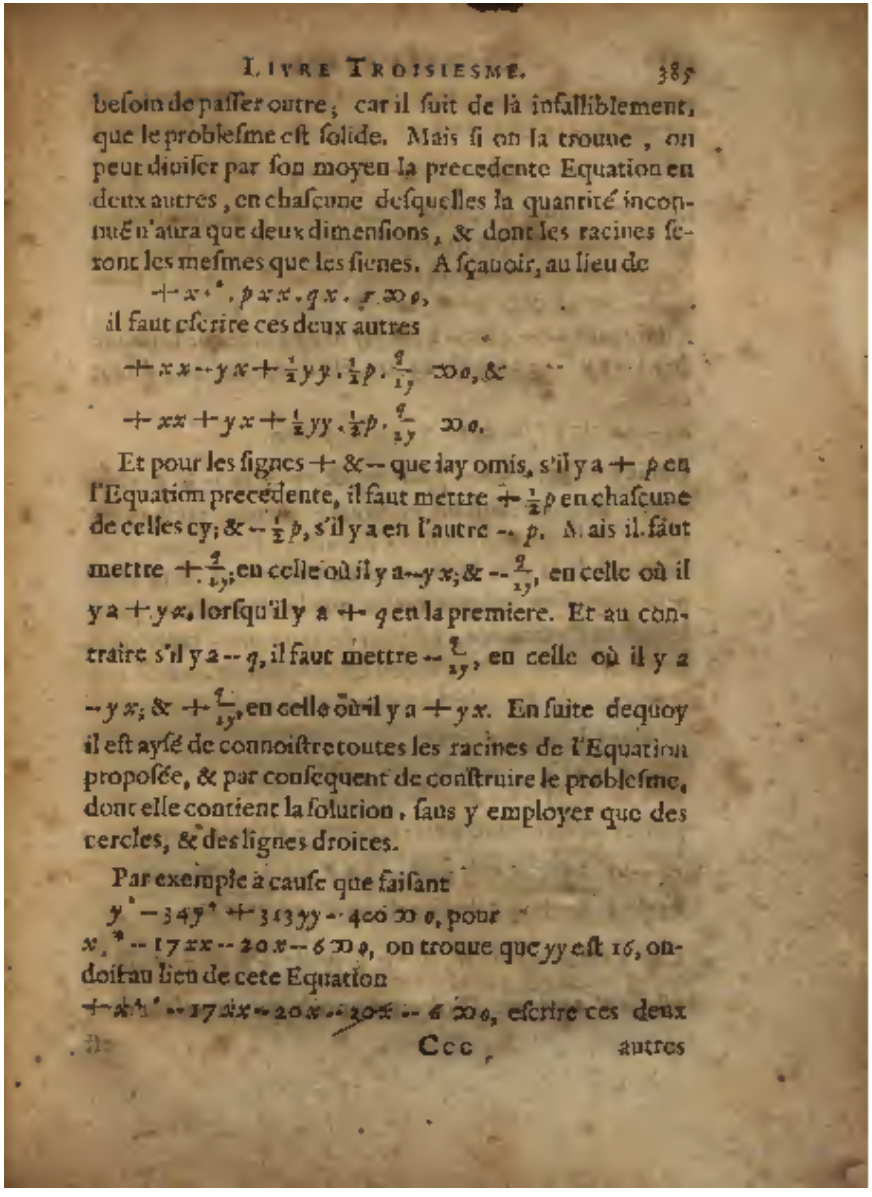
Search for roots of a cubic by examining the factors of the constant term:

if α is such a factor, test whether $x - \alpha$ divides the polynomial.

Examines the example

$$y^6 - 8y^4 - 124y^2 - 64 = 0$$

Descartes on quartics



LIVRE TROISIESME. 385

besoin de passer outre; car il suit de là infalliblement, que le problesme est solide. Mais si on la trouue, on peut diuiser par son moyen la precedente Equation en deux autres, en chascune desquelles la quantité inconnuë n'aura que deux dimensions, & dont les racines seront les mesmes que les siennes. A sçauoir, au lieu de

$+x^4.pxx.qx.r \propto 0$,

il faut escrire ces deux autres

$+xx-yx+\frac{1}{2}yy.\frac{1}{2}p.\frac{q}{2y} \propto 0$, &

$+xx+yx+\frac{1}{2}yy.\frac{1}{2}p.\frac{q}{2y} \propto 0$.

Et pour les signes + & − que i'ay omis, s'il y a + p en l'Equation precedente, il faut mettre $+\frac{1}{2}p$ en chascune de celles cy; & $-\frac{1}{2}p$, s'il y a en l'autre $-p$. Mais il faut mettre $+\frac{q}{2y}$, en celle où il y a $-yx$; & $-\frac{q}{2y}$, en celle où il y a $+yx$, lorsqu'il y a + q en la premiere. Et au contraire s'il y a $-q$, il faut mettre $-\frac{q}{2y}$, en celle où il y a $-yx$; & $+\frac{q}{2y}$, en celle où il y a $+yx$. En suite dequoy il est aysé de connoistre toutes les racines de l'Equation proposée, & par consequent de construire le problesme, dont elle contient la solution, sans y employer que des cercles, & des lignes droites.

Par exemple à cause que faisant

$y^6-34y^4+343yy-400 \propto 0$, pour

$x^4-17xx-20x-6 \propto 0$, on trouue que yy est 16, on doit au lieu de cete Equation

$+x^4-17xx-20x-30x-6 \propto 0$, escrire ces deux

To solve $+x^4 \star .pxx.qx.r = 0$ (Descartes' notation), that is,

$$x^4 \pm pxx \pm qx \pm r = 0\,,$$

he sought to write the quartic as a product of two quadratics. This led him to

$$y^6 \pm 2py^4 + (pp \pm 4r)yy - qq = 0$$

As in Ferrari's/Cardano's method: a quartic is reduced to a cubic

Summary and a glance ahead

By 1600, general solutions were available for quadratic, cubic and quartic equations — specifically, general solutions in radicals, i.e., solutions constructed from the coefficients of a given polynomial equation via $+$, $-$, $\times$, $\div$, $\sqrt{}$, $\sqrt[3]{}$, $\sqrt[4]{}$, $\ldots$

NB: A solution in radicals may be constructed by ruler and compass.

Spoiler: the general quintic equation is not solvable in radicals.

By the 1770s, mathematicians (notably Lagrange) had come to suspect this, but it was not proved until the 1820s.

So did anything interesting happen in algebra during the 17th and 18th centuries?

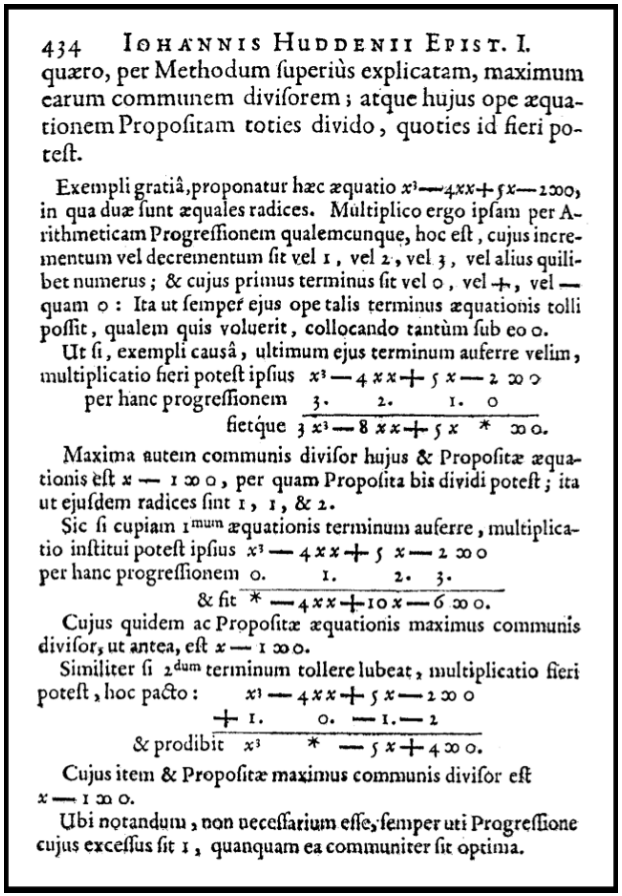
A typical 20th-century view

Luboš Nový, *Origins of modern algebra* (1973), p. 23:

> *From the propagation of Descartes' algebraic knowledge up to the publication of the important works of Lagrange [and others] in the years 1770–1, the evolution of algebra was, at first glance, hardly dramatic and one would seek in vain for great and significant works of science and substantial changes.*

Fair point? Or not?

Some 17th-century developments: Hudde's rule (1657)



434 IOHANNIS HUDDENII EPIST. I.

quæro, per Methodum ſuperiùs explicatam, maximum earum communem diviſorem; atque hujus ope æquationem Propoſitam toties divido, quoties id fieri poteſt.

Exempli gratiâ, proponatur hæc æquatio $x^3 - 4xx + 5x - 2 \infty 0$, in qua duæ ſunt æquales radices. Multiplico ergo ipſam per Arithmeticam Progreſſionem qualemcunque, hoc eſt, cujus incrementum vel decrementum ſit vel 1, vel 2, vel 3, vel alius quilibet numerus; & cujus primus terminus ſit vel 0, vel +, vel — quam 0: Ita ut ſemper ejus ope talis terminus æquationis tolli poſſit, qualem quis voluerit, collocando tantùm ſub eo 0.

Ut ſi, exempli causâ, ultimum ejus terminum auferre velim, multiplicatio fieri poteſt ipſius $x^3 - 4xx + 5x - 2 \infty 0$
per hanc progreſſionem 3. 2. 1. 0
fietque $3x^3 - 8xx + 5x * \infty 0$.

Maxima autem communis diviſor hujus & Propoſitæ æquationis eſt $x - 1 \infty 0$, per quam Propoſita bis dividi poteſt; ita ut ejuſdem radices ſint 1, 1, & 2.

Sic ſi cupiam 1^mum æquationis terminum auferre, multiplicatio inſtitui poteſt ipſius $x^3 - 4xx + 5x - 2 \infty 0$
per hanc progreſſionem 0. 1. 2. 3.
& fit $* - 4xx + 10x - 6 \infty 0$.

Cujus quidem ac Propoſitæ æquationis maximus communis diviſor, ut antea, eſt $x - 1 \infty 0$.

Similiter ſi 2^dum terminum tollere lubeat, multiplicatio fieri poteſt, hoc pacto: $x^3 - 4xx + 5x - 2 \infty 0$
$+1. \quad 0. \quad -1. \quad -2$
& prodibit $x^3 * - 5x + 4 \infty 0$.

Cujus item & Propoſitæ maximus communis diviſor eſt $x - 1 \infty 0$.

Ubi notandum, non neceſſarium eſſe, ſemper uti Progreſſione cujus exceſſus ſit 1, quanquam ea communiter ſit optima.

Published 1659 as an addendum to van Schooten's Latin translation of Descartes' *La géométrie*:

take $x^3 - 4xx + 5x - 2 = 0$

multiply the terms of the equation by numbers in arithmetic progression:

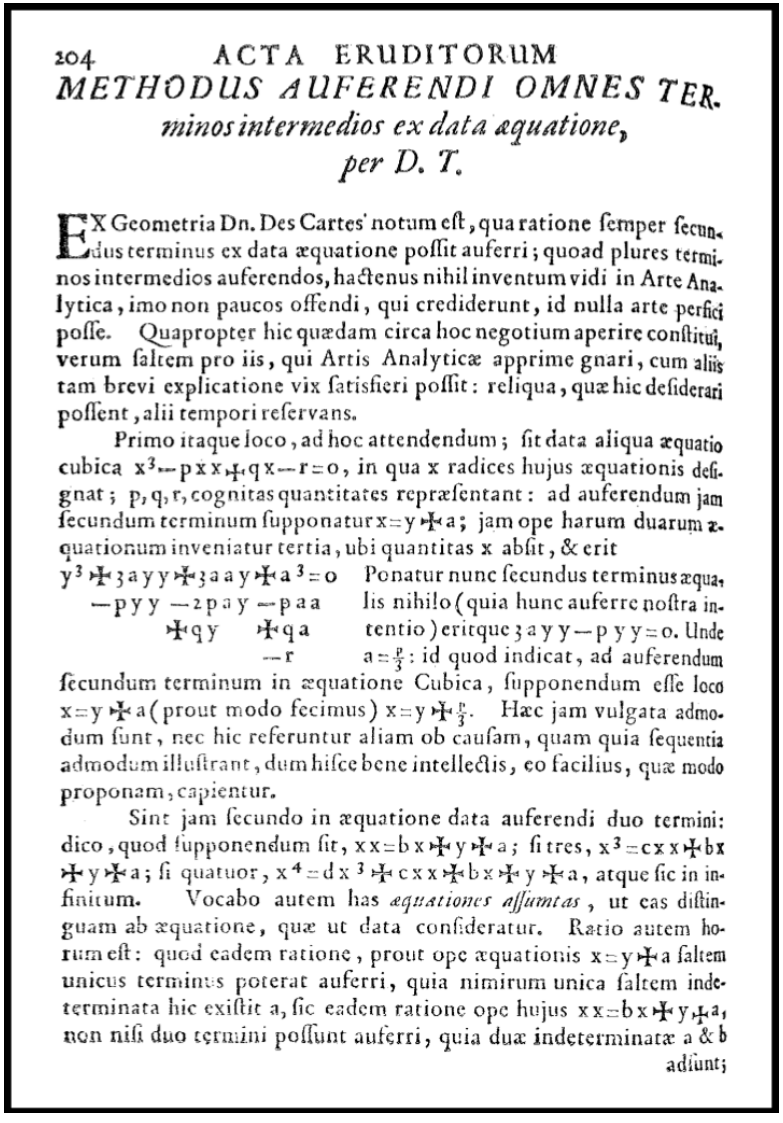
$3x^3 - 8xx + 5x = 0$

$\gcd(x^3 - 4xx + 5x - 2, 3x^3 - 8xx + 5x) = x - 1$

so the original polynomial can be divided by $x - 1$ twice — hence it has a double root at $x = 1$

See *Mathematics emerging*, §12.2.2.

Some 17th-century developments: Tschirnhaus transformations (1683)



204 ACTA ERUDITORUM

METHODUS AUFERENDI OMNES TERminos intermedios ex data æquatione, per D. T.

EX Geometria Dn. Des Cartes notum eſt, qua ratione ſemper ſecundus terminus ex data æquatione poſſit auferri; quoad plures terminos intermedios auferendos, hactenus nihil inventum vidi in Arte Analytica, imo non paucos offendi, qui crediderunt, id nulla arte perfici poſſe. Quapropter hic quædam circa hoc negotium aperire conſtitui, verum ſaltem pro iis, qui Artis Analyticæ apprime gnari, cum aliis tam brevi explicatione vix ſatisfieri poſſit: reliqua, quæ hic deſiderari poſſent, alii tempori reſervans.

Primo itaque loco, ad hoc attendendum; ſit data aliqua æquatio cubica $x^3 - pxx + qx - r = o$, in qua x radices hujus æquationis deſignat; p, q, r, cognitas quantitates repræſentant: ad auferendum jam ſecundum terminum ſupponatur $x = y + a$; jam ope harum duarum æquationum inveniatur tertia, ubi quantitas x abſit, & erit

$y^3 + 3ayy + 3aay + a^3 = o$
$-pyy - 2pay - paa$
$+qy + qa$
$-r$

Ponatur nunc ſecundus terminus æqualis nihilo (quia hunc auferre noſtra intentio) eritque $3ayy - pyy = o$. Unde $a = \frac{p}{3}$: id quod indicat, ad auferendum ſecundum terminum in æquatione Cubica, ſupponendum eſſe loco $x = y + a$ (prout modo fecimus) $x = y + \frac{p}{3}$. Hæc jam vulgata admodum ſunt, nec hic referuntur aliam ob cauſam, quam quia ſequentia admodum illuſtrant, dum hiſce bene intellectis, eo facilius, quæ modo proponam, capientur.

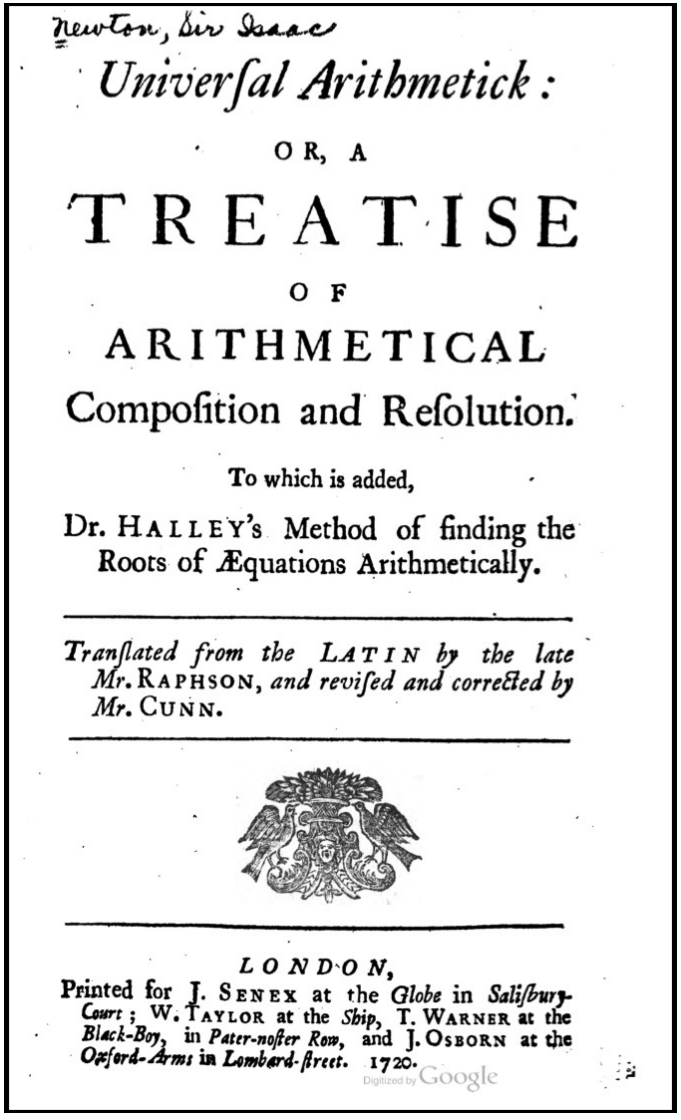
Sint jam ſecundo in æquatione data auferendi duo termini: dico, quod ſupponendum ſit, $xx = bx + y + a$; ſi tres, $x^3 = cxx + bx + y + a$; ſi quatuor, $x^4 = dx^3 + cxx + bx + y + a$, atque ſic in infinitum. Vocabo autem has *æquationes aſſumtas*, ut eas diſtinguam ab æquatione, quæ ut data conſideratur. Ratio autem horum eſt: quod eadem ratione, prout ope æquationis $x = y + a$ ſaltem unicus terminus poterat auferri, quia nimirum unica ſaltem indeterminata hic exiſtit a, ſic eadem ratione ope hujus $xx = bx + y + a$, non niſi duo termini poſſunt auferri, quia duæ indeterminatæ a & b adſunt;

For an equation $x^3 - px^2 + qx - r = 0$

- to remove one term put $x = y + a$ (where $a = p/3$)
- can we remove both the middle terms?
- to remove two terms put $x^2 = bx + y + a$

See *Mathematics emerging*, §12.2.3.

An 18th-century development: Newton's *Arithmetica universalis* (1707)



Newton, Sir Isaac

Universal Arithmetick:

OR, A

TREATISE

OF

ARITHMETICAL

Composition and Resolution.

To which is added,

Dr. HALLEY's Method of finding the Roots of Æquations Arithmetically.

Translated from the LATIN by the late Mr. RAPHSON, and revised and corrected by Mr. CUNN.

LONDON,

Printed for J. SENEX at the *Globe* in *Salisbury-Court*; W. TAYLOR at the *Ship*, T. WARNER at the *Black-Boy*, in *Pater-noster Row*, and J. OSBORN at the *Oxford-Arms* in *Lombard-street*. 1720.

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Rules for sums of powers of roots of

$$x^n - px^{n-1} + qx^{n-2} - rx^{n-3} + sx^{n-4} - \cdots = 0$$

$$\begin{aligned}
\text{sum of roots} &= p \\
\text{sum of roots}^2 &= pa - 2q \\
\text{sum of roots}^3 &= pb - qa + 3r \\
\text{sum of roots}^4 &= pc - qb + ra - 4s
\end{aligned}$$

Developments of the 17th and 18th centuries

- Symbolic notation
- Understanding of the structure of polynomials
- . . . of the number and nature of their roots
- . . . of the relationship between roots and coefficients
- . . . of how to manipulate them
- . . . of how to solve them numerically
- The leaving behind of geometric intuition?

Some 18th-century theory of equations

Recall:

- cubic equations can be solved by means of quadratics
- quartic equations can be solved by means of cubics

Some 18th-century theory of equations

Recall:

- quadratic equations can be solved by means of linear equations
- cubic equations can be solved by means of quadratics
- quartic equations can be solved by means of cubics

The ‘reduced’ or ‘resolvent’ equation:

- for cubics, the reduced equation is of degree 2
- for quartics, the reduced equation is of degree 3
- for quintics, the reduced equation is of degree ?

Some 18th-century hypotheses

Euler's hypothesis (1733):

- for an equation of degree n the degree of the reduced equation will be $n - 1$

Bézout's hypothesis (1764):

- for an equation of degree n the degree of the reduced equation will in general be $n!$
- though always reducible to $(n - 1)!$
- possibly further reducible to $(n - 2)!$

Lagrange's 'Réflexions' 1770/71

J.-L. Lagrange, 'Réflexions sur la résolution algébrique des équations', Berlin (1770/1):

Examined all known methods of solving

- quadratics: the well-known solution
- cubics: methods of Cardano, Tschirnhaus, Euler, Bézout
- quartics: methods of Cardano, Descartes, Tschirnhaus, Euler, Bézout

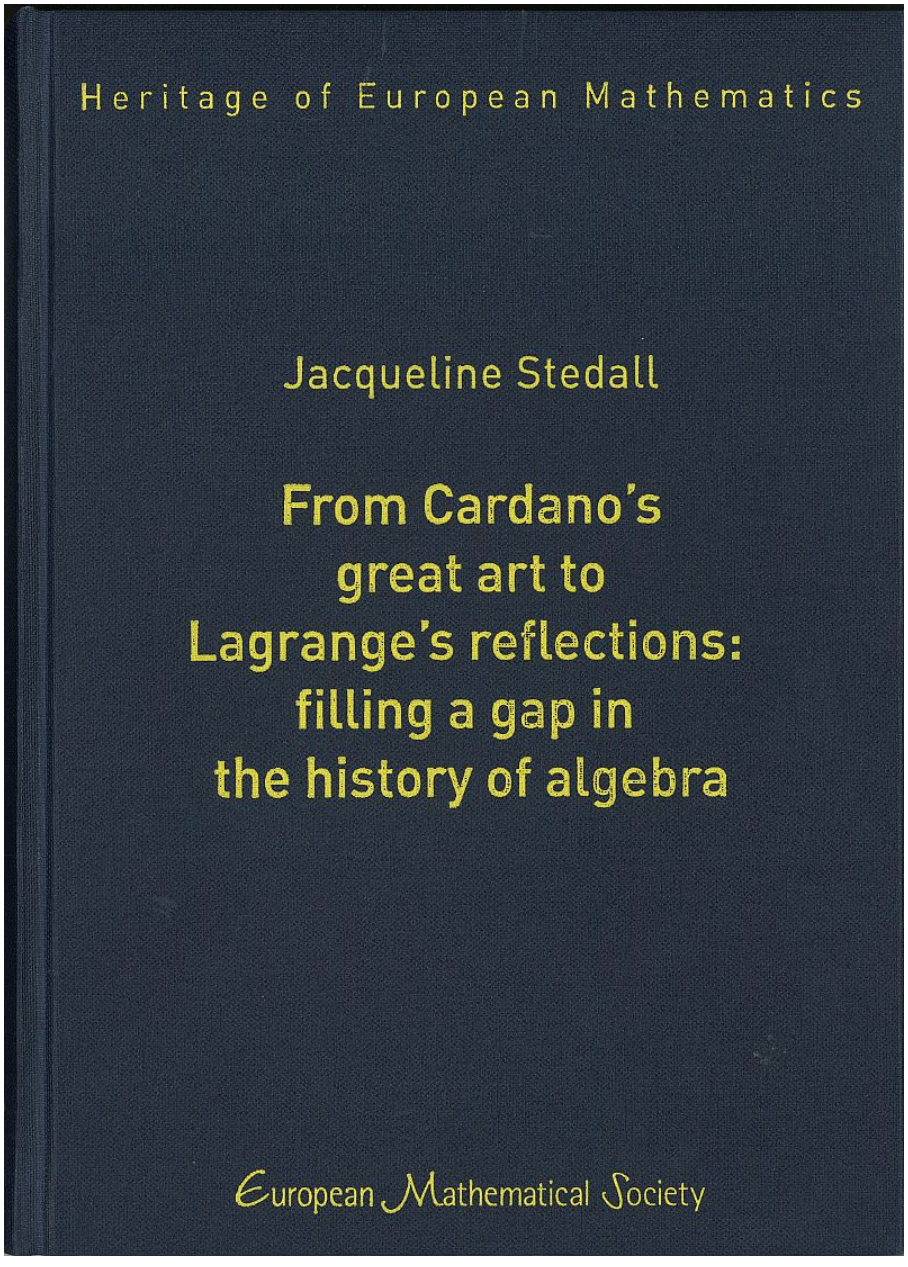
seeking to identify a uniform method that could be extended to higher degree

A typical 20th-century view revisited

Luboš Nový, *Origins of modern algebra* (1973), p. 23:

> *From the propagation of Descartes' algebraic knowledge up to the publication of the important works of Lagrange [and others] in the years 1770–1, the evolution of algebra was, at first glance, hardly dramatic and one would seek in vain for great and significant works of science and substantial changes.*

Filling a gap in the history of algebra (2011)


Heritage of European Mathematics
Jacqueline Stedall
From Cardano's great art to Lagrange's reflections: filling a gap in the history of algebra
European Mathematical Society

The hitherto untold story of the slow and halting journey from Cardano's solution recipes to Lagrange's sophisticated considerations of permutations and functions of the roots of equations . . . [Preface]

From Stedall's preface:

This assertion ... from Nový quoted above, betrays yet another fundamental shortcoming of several earlier accounts, a view that mathematics somehow progresses only by means of 'great and significant works' and 'substantial changes'. Fortunately, the truth is far more subtle and far more interesting: mathematics is the result of a cumulative endeavour to which many people have contributed, and not only through their successes but through half-formed thoughts, tentative proposals, partially worked solutions, and even outright failure. No part of mathematics came to birth in the form that it now appears in a modern textbook: mathematical creativity can be slow, sometimes messy, often frustrating.

BO1 History of Mathematics
Lecture X
The 19th-century beginnings of 'modern algebra'

MT 2022 Week 5

Summary

- Lagrange's ideas (1770/71)
- Cauchy and substitutions (1815)
- 'Classical age' of theory of equations 'ends' (1799–1826)
- The invention of groups by Galois and Cauchy
- 'Symbolical algebra'
- Groups, rings, and fields: the emergence of 'modern algebra' (1854–1900)

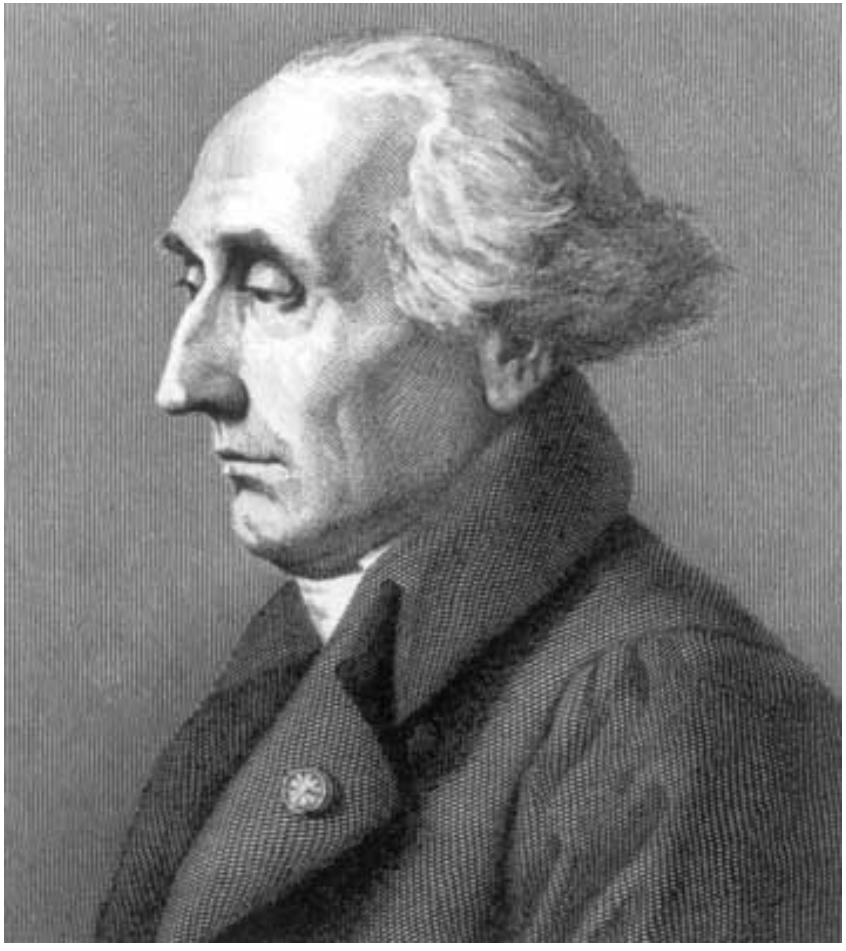
‘Modern’ or ‘abstract’ algebra

19th century: emergence of mathematics whose subject-matter is no longer space or number:

- permutations
- abstract structures (groups, rings, fields, ...)
- linear algebra [see Lecture XIV]

Lagrange's 'Réflexions' 1770/71

J.-L. Lagrange, 'Réflexions sur la résolution algébrique des équations', Berlin (1770/71):



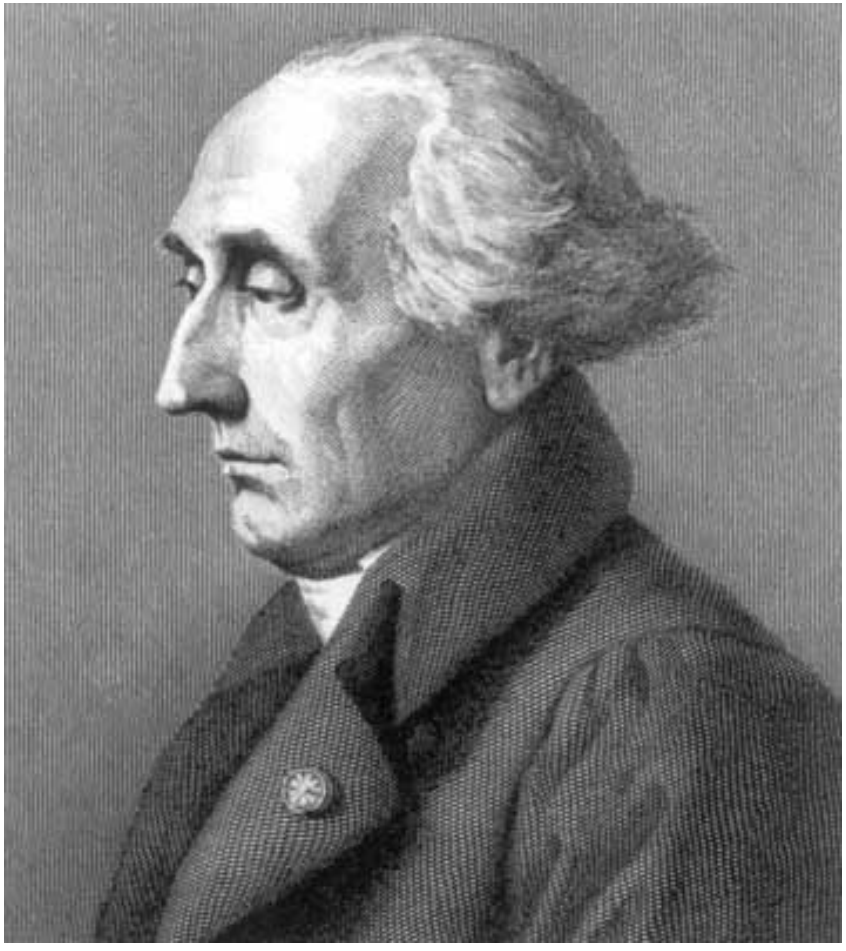
Lagrange's 'Réflexions' 1770/71

J.-L. Lagrange, 'Réflexions sur la résolution algébrique des équations', Berlin (1770/71):

Asserted that there had been little advance in equation-solving since Cardano, but that there was little left to do

Examined all known methods of solving cubics and quartics

Found that in every case the solutions of the 'reduced' (or 'resolvent') equation are 'functions' of the roots of the equation to be solved



Resolvents for cubics

For a cubic with roots x_1, x_2, x_3 there is a reduced equation whose roots are values of

$$y = \frac{1}{3}\left(x_1 + \alpha^2 x_2 + \alpha x_3\right)$$

where $\alpha^3 = 1$, $\alpha \neq 1$.

Lagrange: since y^3 takes just 2 values as x_1, x_2, x_3 are permuted, it satisfies a quadratic equation: the resolvent of the cubic

Lagrange identified this idea as a feature common to the methods for solving cubics presented by Cardano, Tschirnhaus, Bézout, and Euler

Resolvents for quartics

For a quartic with roots x_1, x_2, x_3, x_4 there is a reduced equation whose roots are values of

$$y = \frac{1}{2}(x_1x_2 + x_3x_4)$$

Lagrange: since y takes just 3 values as x_1, x_2, x_3, x_4 are permuted, it satisfies a cubic equation.

There is also reduced equation whose roots are values of

$$z = \frac{1}{2}[(x_1 + x_2) - (x_3 + x_4)]$$

Lagrange: since z^2 takes just 3 values as x_1, x_2, x_3, x_4 are permuted, it satisfies a cubic equation.

Resolvents in general

Let the given equation be of degree n with roots
$x_1, x_2, x_3, \cdots, x_n$

Theorem: Let $y = f(x_1, x_2, x_3, \cdots, x_n)$. Then y is a root of an equation of degree m, where m is the number of values taken by y (that is, by f) under permutations of $x_1, x_2, x_3, \cdots, x_n$

Theorem: The number of values of f will always be a divisor of $n!$

Note 1: The insight is wonderful; the proof is not

Note 2: This theorem mutated several times, finally morphing into Lagrange's Theorem in group theory.

(See *Mathematics emerging*, §12.3.1, and also: Richard L. Roth, A History of Lagrange's Theorem on Groups, *Mathematics Magazine* **74**(2) (2001), pp. 99–108)

Resolvents for equations of degree 5

For a quintic equation the general resolvent equation will be of degree 120. But if

$$y := x_1 + \varphi^4 x_2 + \varphi^3 x_3 + \varphi^2 x_4 + \varphi x_5$$

then y^5 takes only 24 values, so satisfies a resolvent equation of degree 24.

This can be reduced to another resolvent equation of degree 6.

Is there any hope of reducing it further?

Those influenced by Lagrange: Paolo Ruffini

Paolo Ruffini (1765–1822), *Teoria generale delle equazione* (1799) and a number of articles 1804–1819:

- showed that a function of 5 variables cannot take 3 or 4 values
- claimed to have proved that equations of degree 5 were not in general solvable by radicals
- sent book to Lagrange in 1802 and to Paris academy; no response
- further explanatory papers 1802, 1806, ...

A long, confused and confusing account, which seems to have persuaded no-one except Italian pupils and colleagues?

Those influenced by Lagrange: A.-L. Cauchy

A.-L. Cauchy (1789–1857), 'Mémoire sur le nombre de valeurs qu'une fonction peut acquérir, lorsqu'on y permute de toutes les manières possibles les quantités qu'elle renferme', *Journal de l'École polytechnique*, 1815:

Established notation and some theory for substitutions

Theorem: Let N be the number of values of a function of n variables. Either $N \leq 2$ or $N \geq p$ for any prime number $p \leq n$.

Conjecture: For $n \geq 5$, either $N \leq 2$ or $N \geq n$.

Proved his conjecture for $n = 6$

(See *Mathematics emerging*, §13.1.1.)

Those influenced by Lagrange: N. H. Abel

Niels Henrik Abel (1802–1829), 'Beweis der Unmöglichkeit, algebraische Gleichungen von höheren Graden als dem vierten allgemein aufzulösen', *Crelle's Journal*, 1826:

For $n = 5$, refined Cauchy's 1815 theorem

Used this to prove conclusively that the general equation of degree 5 is not soluble by radicals.

That is, there is no formula involving radicals for a solution of equations of degree 5.

The end of 'classical algebra' (?)

Évariste Galois (1811–1832)

Galois and his groups (1)

Évariste Galois (1811–1832), 'Mémoire sur les conditions de résolubilité des équations par radicaux', manuscript known as the *Premier mémoire*

Explored the question of which numerical equations are soluble by radicals, which not:

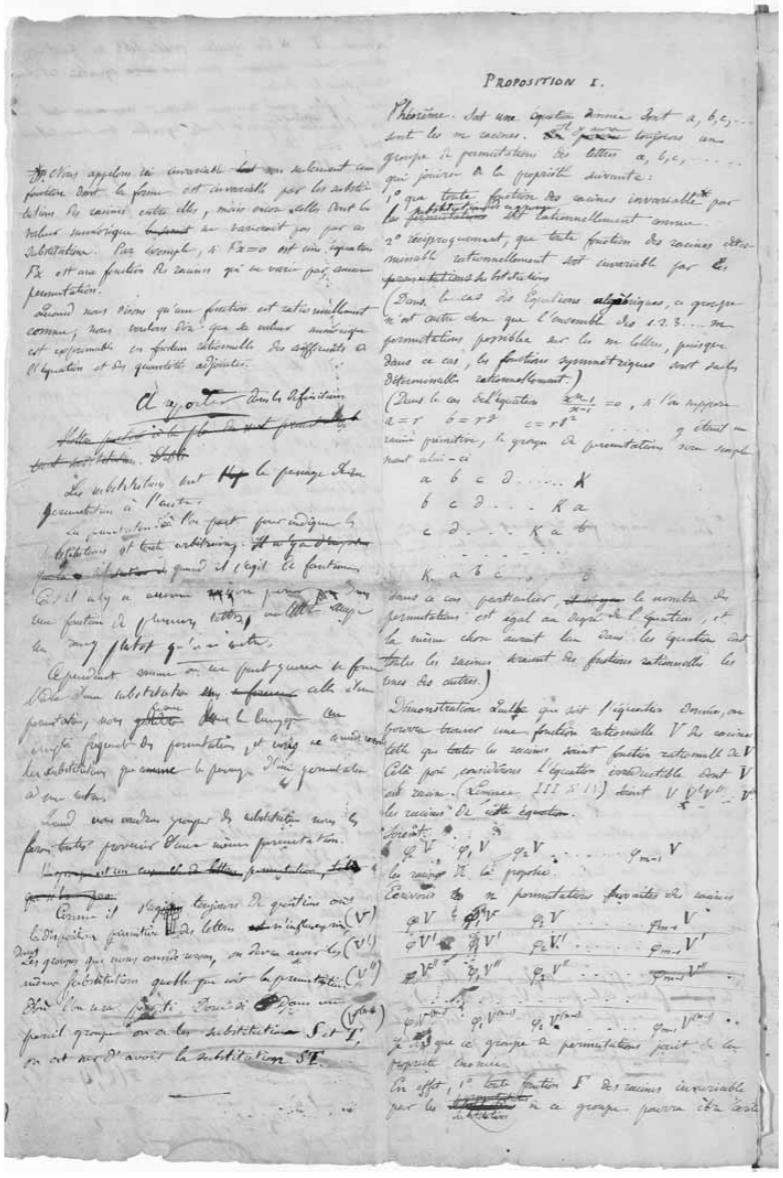
- submitted his ideas to the Academy in 1829, withdrew his articles January 1830 on Cauchy's advice
- resubmitted February 1830; lost after Fourier died in 1830
- resubmitted January 1831, rejected by Academy on Poisson's advice in July 1831
- corrected by Galois up to his death by duel in 1832
- to be read in conjunction with Galois' Testamentary Letter of 29 May 1832 to Auguste Chevalier

Galois and his groups (2)

Galois, in his writings 1829/30 (published 1846):

- invented groups (of permutations) [note: Cauchy invented groups in 1845, almost certainly independently]
- pre-invented fields (in 'Théorie des nombres' [published 1830] and as his 'rationally known quantities')
- showed how to associate a group to a polynomial (its Galois group)
- discovered a necessary and sufficient condition for solubility of an equation by radicals expressed in terms of the structure of its group
- as an application, gave a necessary and sufficient condition for solubility of an irreducible equation of prime degree by radicals

Galois and his groups (3)



PROPOSITION I.

Premier mémoire, dossier 1, folio 3 verso

Proposition I relates a given polynomial to a group of permutations (its Galois group)

Eleventh-hour marginal additions provide further explanation

Galois and his groups (4)

The eleventh-hour marginal addition in translation:

> *Substitutions are the passage from one permutation to another.*
>
> *The permutation from which one starts in order to indicate substitutions is completely arbitrary, ...*
>
> *... one must have the same substitutions, whichever permutation it is from which one starts. Therefore, if in such a group one has substitutions S and T, one is sure to have the substitution ST.*

Évariste Galois, 29th May 1832, published 1846

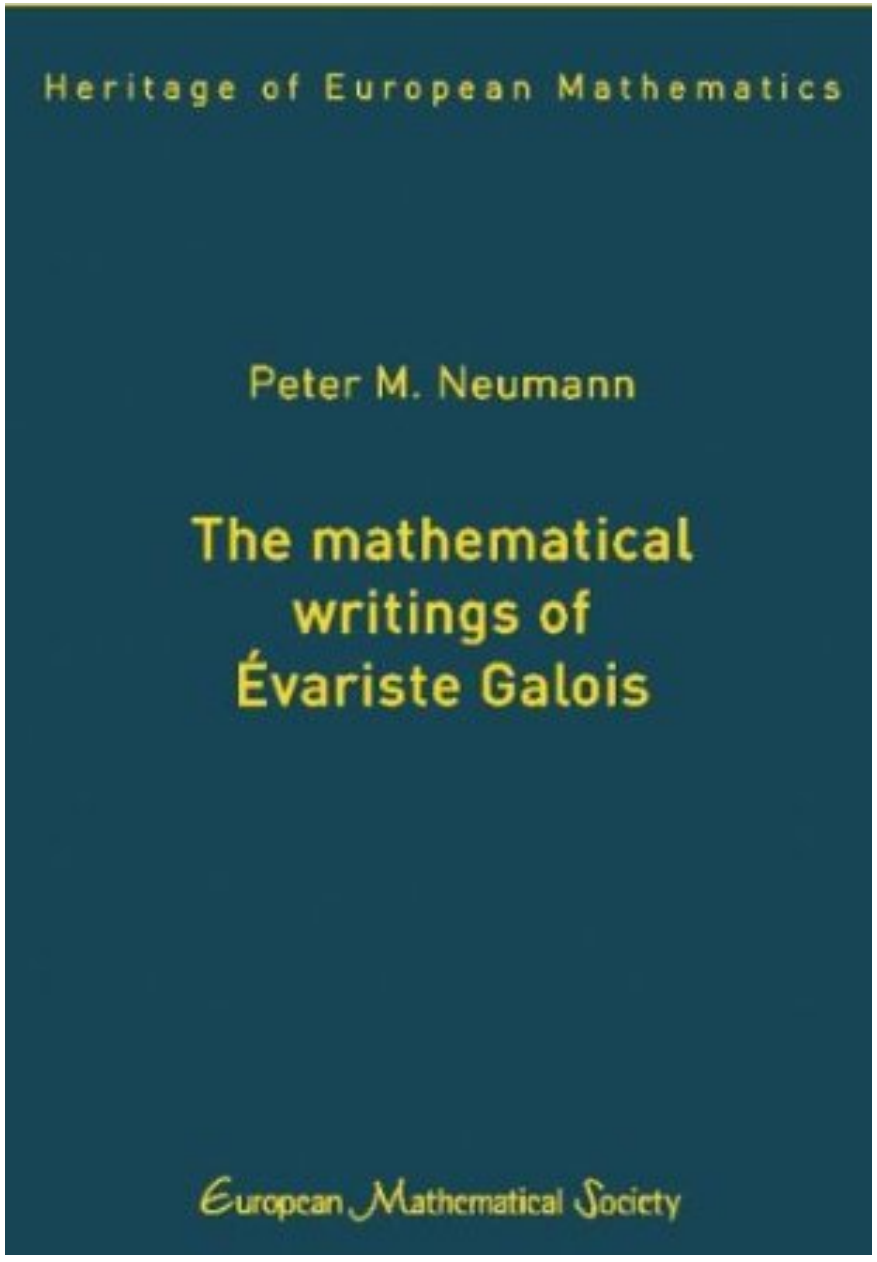
(See *Mathematics emerging*, §13.1.2.)

Galois and his groups: publication

Publication of Galois' results:

1829–30:	5 articles inc. 'Sur la théorie des nombres'
1832:	Testamentary Letter to Chevalier
1846:	letter and all other major papers by Liouville
1897:	Liouville's edition re-published by Picard
1906/07/08:	minor manuscripts published by Tannery
1962:	complete Bourgne & Azra edition
25 Oct 2011:	English/French bilingual edition by Peter Neumann

Galois in English (2011)


Heritage of European Mathematics
Peter M. Neumann
The mathematical writings of Évariste Galois
European Mathematical Society

Cauchy and his 'groups' (1)

Meanwhile, in 1845 ...

Joseph Bertrand proved Cauchy's conjecture from 1815 (subject to a Postulate) ...

and submitted a paper to the Paris Academy in March 1845;

in April 1845, Cauchy was given Bertrand's paper for review ...

and from September 1845 to January 1846 published a stream of papers on the same topic (and introducing a version of groups),

giving his report on Bertrand's paper in November 1845,

which was eventually published in November 1848

(Peter M. Neumann, 'On the date of Cauchy's contributions to the founding of the theory of groups', *Bull. Austral. Math. Soc.* **40** (1989), 293–302.)

Cauchy and his 'groups' (2)

Cauchy's definition of a 'group' (1845):

Consider substitutions $\binom{A}{B}$, $\binom{C}{D}$, $\binom{E}{F}, \ldots$ and all those derived from them by multiplying them together one or more times in any order. These form a système de substitutions conjuguées (a system of conjoined substitutions).

His purpose: for any function $f(x_1, x_2, \ldots, x_n)$ the substitutions that leave it unchanged (yielding 'valeurs égales') form such a system. The number of values of the function ('valeurs différentes') is the index of this system — that is $\frac{n!}{N}$, where N is the number of its members.

Hence — a proof of his 1815 conjecture and more.

The Paris Grand Prix of 1860

Académie des Sciences, Paris, *Grand Prix de Mathématiques*, 1860: subject announced April 1857 (Cauchy on committee, he died a month later):

> *What are the possibilities for the number of values of well defined functions containing a given number of letters, and how can one form the functions for which there exist a given number of values?*

Grand Prix 1860: responses

Three competitors:

- Émile Mathieu;
- Camille Jordan
  (submitted their Paris doctoral dissertations);
- Rev. Thomas Penyngton Kirkman
  (submitted his essay 'The complete theory of groups').

None successful.

All interpreted the problem as 'find all subgroups of Sym(n)'.

The competition stimulated:

- development of theory of (finite permutation) groups;
- merger of Galois' and Cauchy's independent theories.

Meanwhile, in Britain...

A sideline in the development of algebra in the 19th century was the notion of symbolical algebra that emerged in Britain c. 1830

This was a response to an argument advanced by a (persistent) minority of British mathematicians that the notion of a negative number is invalid

It first appeared in print in the 1830 *A Treatise of Algebra* by George Peacock (1791–1858)

Symbolical algebra

In symbolical algebra, symbols are regarded initially as being without interpretation, but may be manipulated according to specified rules; interpretation comes at the end of the process.

This approach was not accepted by all mathematicians in Britain, and sparked a debate about the nature of mathematical truth: can operation really precede interpretation?

Contributors to symbolical algebra included George Peacock (1791–1858), Duncan Gregory (1813–1844), Augustus De Morgan (1806–1871), George Boole (1815–1864), and William Rowan Hamilton (1805–1865).

But the idea of symbolical algebra had largely faded away by the middle of the century: one *could* work with arbitrary operations in an entirely abstract setting, but why would one want to?

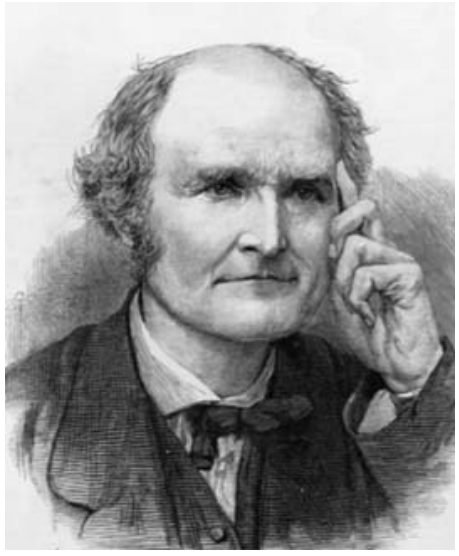
Cayley and his groups (1)

Arthur Cayley, 'On the theory of groups, as depending on the symbolic equation $\theta^n = 1$' (1854):

A set of symbols

$$1, \alpha, \beta, \ldots$$

all of them different, and such that the product of any two of them (no matter in what order), or the product of any one of them into itself belongs to the set, is said to be a group.



Cayley widely attributed with introducing 'abstract' theory of groups

(See *Mathematics emerging*, §13.1.4.)

Cayley and his groups (2)

Examples of groups of order 4:

- roots of $x^4 - 1 = 0$
- other examples from elliptic functions, quadratic forms
- matrices $(A, A^{-1}, A^T, (A^T)^{-1})$

Examples of groups of order 6:

- permutations of three letters
- operations from his previous paper 'properties of a caustic'

Key question for Cayley later: how many groups of order n?

Cayley, 'On the theory of groups' (1878):

> *A group is defined by means of the laws of combinations of its symbols.*

Weber's axioms, 1882

A System G of h elements of any kind, $\Theta_1, \Theta_2, \ldots, \Theta_h$ is called a group of degree h, if it satisfies the following conditions:

I. By some rule, which will be called composition or multiplication, from two elements of the system a new element of the system may be derived. In symbols:

$$\Theta_r \Theta_s = \Theta_t.$$

II. Always:

$$(\Theta_r \Theta_s)\Theta_t = \Theta_r(\Theta_s \Theta_t) = \Theta_r \Theta_s \Theta_t.$$

III. From $\Theta\Theta_r = \Theta\Theta_s$ and from $\Theta_r\Theta = \Theta_s\Theta$ follows $\Theta_r = \Theta_s$.

Existence of identity and inverses appear as deductions from the axioms — incorporated as axioms by later authors

On axiomatisation of groups

Peter M. Neumann, 'What groups were: a study of the development of the axiomatics of group theory', *Bull. Austral. Math. Soc.* **60** (1999), 285–301.

Christopher D. Hollings, '"Nobody could possibly misunderstand what a group is": a study in early twentieth-century group axiomatics', *Arch. Hist. Exact Sci.* **71**(5) (2017), 409–481.

Rings and ideals

Ernst Kummer (1844):

- concerned with Fermat's last theorem, and quadratic forms
- worked with arithmetic of 'cyclotomic integers' $a_0 + a_1\theta + a_2\theta^2 + \cdots + a_{n-1}\theta^{n-1}$ where θ is primitive n-th root of 1
- discovered that unique factorisation need not hold
- devised the concept of 'ideal' factors

Richard Dedekind, 'Sur la théorie des nombres entiers algébriques' (1877) and famous appendices to his editions of Dirichlet's *Lectures on Number Theory*:

- changed Kummer's 'ideal numbers' to 'ideals'
- worked also with rings of numbers [domains] and fields of numbers [Körper]

‘Abstract algebra’ begins to form

Specific instances of fields studied by Galois, Kronecker, Dedekind, and others. First axiomatic definition due to Weber, 1893.

Ernst Steinitz, ‘Algebraische Theorie der Körper’, 1910: first comprehensive presentation of the theories of fields, modules, and vector spaces [to be revisited in a later lecture]

Specific rings studied by Dedekind, Hilbert, ... Early axiomatic definition given by Fraenkel in 1914: not quite the same as the modern definition.

Abstract algebra given an early boost (in the USA) via a short-lived obsession with ‘postulate analysis’: the study of systems of axioms for their own sake

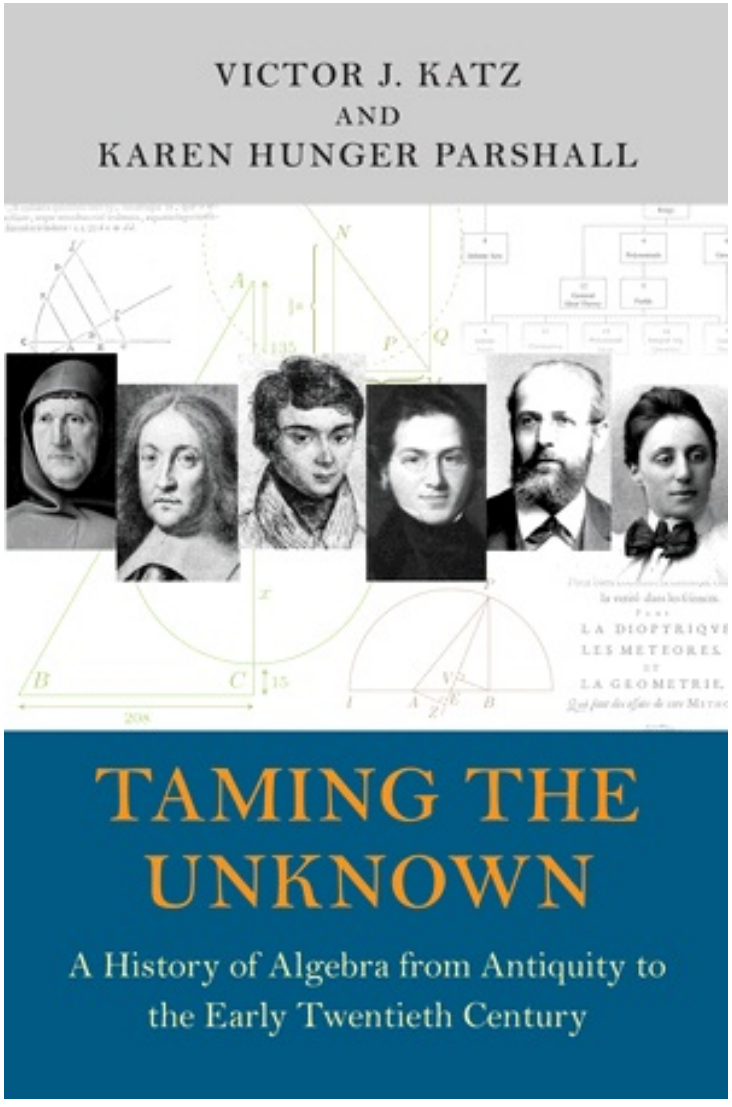
‘Abstract algebra’ takes off

Comprehensive abstract study of (commutative) rings initiated by Emmy Noether in the 1920s, sometimes mirroring the earlier ‘concrete’ work of Dedekind: ‘Es steht alles schon bei Dedekind’.

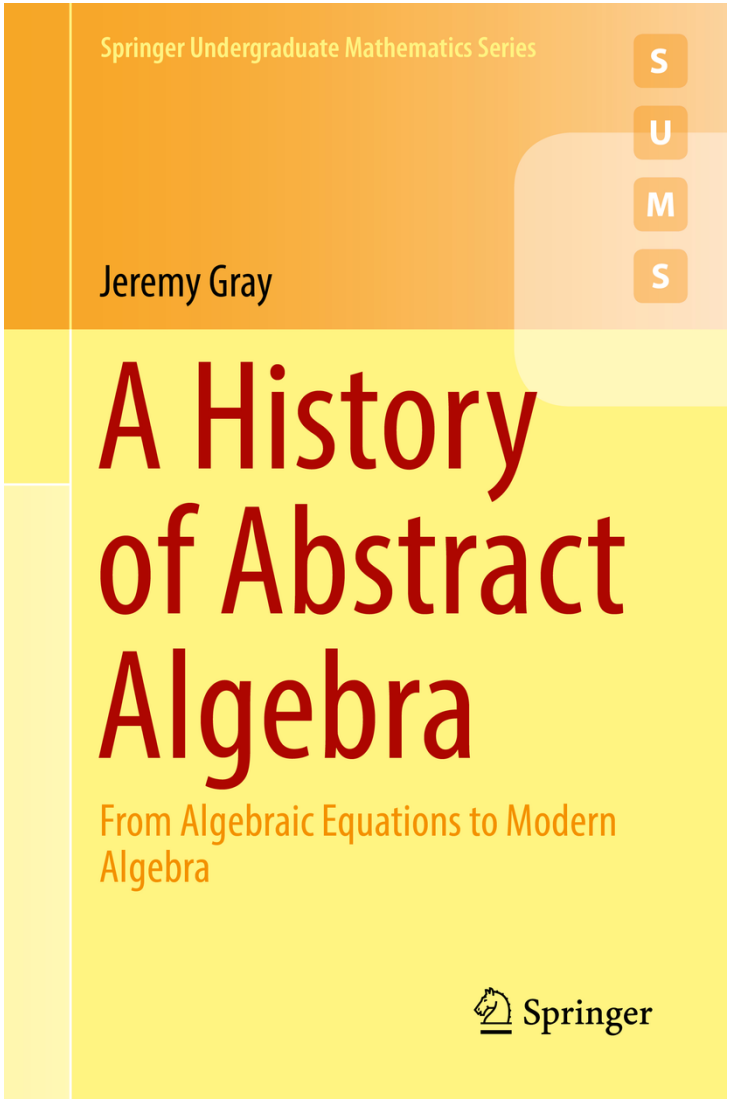
Noether’s lectures (in Göttingen) united with those of Emil Artin (Hamburg) in B. L. van der Waerden’s highly influential textbook *Moderne Algebra* (1930).

Abstract point of view now dominant, with many different objects studied: groups, fields, rings, integral domains, semigroups, algebras, lattices, semirings, quasigroups, ...

Overviews of the topics of lectures IX and X


VICTOR J. KATZ
AND
KAREN HUNGER PARSHALL
TAMING THE UNKNOWN
A History of Algebra from Antiquity to the Early Twentieth Century

(Princeton Univ. Press, 2014)



(Springer, 2018)

BO1 History of Mathematics
Lecture XI
19th-century rigour in real analysis, part 1

MT 2022 Week 6

Summary

- New difficulties emerge
- Continuity and convergence
- Integration
- The Fundamental Theorem of Calculus
- New ideas about integration

Recall from lecture VIII: Fourier series, 1822

Joseph Fourier, *Théorie analytique de la chaleur* [*Analytic theory of heat*] (1822):

Suppose that $\phi(x) = a \sin x + b \sin 2x + c \sin 3x + \cdots$

and also that $\phi(x) = x\phi'(0) + \frac{1}{6}x^3\phi'''(0) + \cdots$

After many pages of calculations, multiplying and comparing power series, Fourier found that the coefficient of $\sin nx$ must be

$$\frac{2}{\pi} \int_0^\pi \phi(x) \sin nx \, dx$$

Fourier's derivation was based on 'naive' manipulations of infinite series. It was ingenious but non-rigorous by today's standards.

BUT it led to profound results

New doubts in the early 19th century

Fourier's work converged with more philosophical investigation to stimulate questions concerning:

- functions — what exactly should they be?
- convergence — what exactly should it be?
- convergence of functions — what properties are preserved?
- integration — what exactly should it be?
- existence of limits — what are the essential properties of real numbers? [Lecture XII]

Recall from Lecture VIII: Cauchy sequences, 1821

Augustin-Louis Cauchy, *Cours d'analyse* (1821), Ch. VI, pp. 124, 125:

> *In order for the series* $u_0,\ u_1,\ u_2, \ldots$ [that is, $\sum u_i$] *to be convergent ... it is necessary and sufficient that the partial sums*
>
> $$s_n = u_0 + u_1 + u_2 + \&c. \ldots + u_{n-1}$$
>
> *converge to a fixed limit* s*: in other words, it is necessary and sufficient that for infinitely large values of the number* n*, the sums*
>
> $$s_n,\ s_{n+1},\ s_{n+2},\ \&c. \ldots$$
>
> *differ from the limit* s*, and consequently from each other, by infinitely small quantities.*

Cauchy and continuity revisited

In *Cours d'analyse*, p. 34, Cauchy defined a function f to be continuous between certain limits if, for each x between those limits, the value of $f(x)$ is unique and finite, and $|f(x+\alpha) - f(\alpha)|$, where α is indefinitely small, decreases indefinitely with α.

In other words (p. 35): for x between the given limits, an infinitely small increase in x produces and infinitely small increase in $f(x)$.

So Cauchy defined continuity on an interval, rather than at a point.

He went on to derive basic results concerning continuous functions: that the composition of two continuous functions is continuous, the Intermediate Value Theorem, etc.

A theorem of Cauchy (1821)

Cauchy, *Cours d'analyse*, pp. 131–132:

> *When the various terms of a series are functions of a variable x, continuous with respect to this variable in the neighbourhood of a particular value for which the series is convergent, the sum s of the series is also, in the neighbourhood of this value, a continuous function of x.*

In other words: a convergent series of continuous functions converges to a continuous function.

Not true!

Cauchy's argument

Cauchy considered a sequence of continuous functions $u_0(x), u_1(x), u_2(x), \ldots$ on a given interval. He supposed that the corresponding series converges to a function $s(x)$. Partial sums are denoted by $s_n(x) = \sum_{j=0}^{n-1} u_n(x)$. The nth remainder term $r_n(x)$ is defined by $s(x) = s_n(x) + r_n(x)$.

Cauchy noted that each s_n is evidently continuous for values of x in the given interval. Suppose that we increase x by an infinitely small quantity α. For all values of n, the corresponding increase in $s_n(x)$ will also be infinitely small. For n very large ('très-considérable'), the increase in $r_n(x)$ becomes 'insensible'. Therefore, the increase in $s(x)$ can only be an infinitely small quantity.

NB. All notation except '$\sum$' is Cauchy's.

Cauchy's argument

est convergente, la somme de cette série est représentée par

$$u_0 + u_1 + u_2 + u_3 + \&c. \ldots,$$

En vertu de cette convention, la valeur du nombre e se trouvera déterminée par l'équation

$$(6) \qquad e = 1 + \frac{1}{1} + \frac{1}{1.2} + \frac{1}{1.2.3} + \frac{1}{1.2.3.4} + \&c. \ldots;$$

et, si l'on considère la progression géométrique

$$1, \quad x, \quad x^2, \quad x^3, \quad \&c. \ldots,$$

on aura, pour des valeurs numériques de x inférieures à l'unité,

$$(7) \qquad 1 + x + x^2 + x^3 + \&c. \ldots = \frac{1}{1-x}.$$

La série

$$u_0, \quad u_1, \quad u_2, \quad u_3, \quad \&c. \ldots$$

étant supposée convergente, si l'on désigne sa somme par s, et par s_n la somme de ses n premiers termes, on trouvera

$$s = u_0 + u_1 + u_2 + \ldots + u_{n-1} + u_n + u_{n+1} + \&c. \ldots$$
$$= s_n + u_n + u_{n+1} + \&c. \ldots,$$

et par suite

$$s - s_n = u_n + u_{n+1} + \&c. \ldots$$

De cette dernière équation il résulte que les quantités

$$u_n, \quad u_{n+1}, \quad u_{n+2}, \quad \&c. \ldots.$$

formeront une nouvelle série convergente dont la somme sera équivalente à $s - s_n$. Si l'on représente cette même somme par r_n, on aura

$$s = s_n + r_n;$$

et r_n sera ce qu'on appelle le *reste* de la série (1) à partir du $n.^{me}$ terme.

Lorsque, les termes de la série (1) renfermant une même variable x, cette série est convergente, et ses différens termes fonctions continues de x, dans le voisinage d'une valeur particulière attribuée à cette variable;

$$s_n, \quad r_n \quad \text{et} \quad s$$

sont encore trois fonctions de la variable x, dont la première est évidemment continue par rapport à x dans le voisinage de la valeur particulière dont il s'agit. Cela posé, considérons les accroissemens que reçoivent ces trois fonctions, lorsqu'on fait croître x d'une quantité infiniment petite α. L'accroissement de s_n sera, pour toutes les valeurs possibles de n, une quantité infiniment petite; et celui de r_n deviendra insensible en même temps que r_n, si l'on attribue à n une valeur très-considérable. Par suite, l'accroissement de la fonction s ne pourra être qu'une quantité infiniment petite. De cette remarque on déduit immédiatement la proposition suivante.

1.er THÉORÈME. *Lorsque les différens termes de la série* (1) *sont des fonctions d'une même variable* x,

A modern counterexample

For each $n \in \mathbb{N}$, define continuous functions f_n by

$$f_n(x) = \begin{cases} -1 & \text{if } x \leq -\frac{1}{n}; \\ nx & \text{if } -\frac{1}{n} \leq x \leq \frac{1}{n}; \\ +1 & \text{if } x \geq \frac{1}{n}. \end{cases}$$

Now set $u_1(x) = f_1(x)$, and define new functions u_n recursively by

$$u_n(x) = f_n(x) - f_{n-1}(x).$$

Notice then that

$$s_n(x) = \sum_{j=1}^{n} u_j(x) = f_n(x).$$

But we see that $s_n \to s$ as $n \to \infty$, where

$$s(x) = \begin{cases} -1 & \text{if } x < 0; \\ 0 & \text{if } x = 0; \\ +1 & \text{if } x > 0, \end{cases}$$

which is discontinuous at $x = 0$.

A modern counterexample

What happens to the remainders $r_n(x) = s(x) - s_n(x)$?

Outside the range $-\frac{1}{n} \leq x \leq \frac{1}{n}$, $r_n(x) = 0$, but inside:

$$r_n(x) = \begin{cases} -1 - nx & \text{if } -\frac{1}{n} \leq x < 0; \\ 0 & \text{if } x = 0; \\ 1 - nx & \text{if } 0 < x \leq \frac{1}{n}. \end{cases}$$

For each x, $r_n(x) \to 0$ as $n \to \infty$, but this does not happen simultaneously for all values of x.

Cauchy's remainders

Cauchy: For n very large, the increase in $r_n(x)$ becomes 'insensible'. But what does this mean?

One of the following modern statements? (Denoting Cauchy's interval by I.)

$$\forall \varepsilon > 0 : \exists N : \forall x \in I : n > N \Rightarrow |r_n(x)| < \varepsilon$$

$$\forall \varepsilon > 0 : \forall x \in I : \exists N : n > N \Rightarrow |r_n(x)| < \varepsilon$$

The second is true for our modern counterexample, but the first is not — so there really is a distinction between the two.

Cauchy clearly didn't make this distinction — but should this really be regarded as a 'mistake'?

Reactions to Cauchy's 'mistake'

Abel (Crelle's Journal, 1826) on the *Cours d'analyse*:

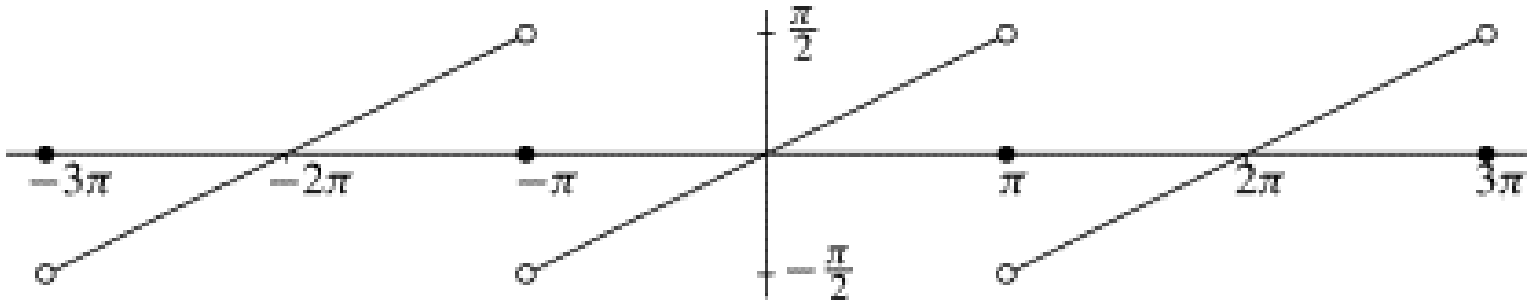
> *the excellent work of M. Cauchy ... which must be read by every analyst who aims at rigour in mathematical research*

Four pages later, on Cauchy's 'theorem' on sums of continuous functions:

> *it seems to me that the theorem admits exceptions. For example, the series*
>
> $$\sin x - \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x - \cdots$$
>
> *is discontinuous for every value* $(2m+1)\pi$ *of* x*,* m *being a whole number. There are, as one knows, many series of this kind.*

Abel's counterexample

$$\sin x - \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x - \cdots$$



Abel's counterexample elsewhere

Abel to Holmboe, January 1826:

> *One applies all operations to infinite series as if they were finite, but is this permissible? I think not. — Where is it proved that one gets the differential of an infinite series by differentiating each term? It is easy to give an example for which this is not true, e.g.*
>
> $$\frac{1}{2}x = \sin x - \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x - \cdots .$$
>
> *Differentiation gives*
>
> $$\frac{1}{2} = \cos x - \cos 2x + \cos 3x - \cdots \text{etc.}$$
>
> *a result which is quite false because this series is divergent.*

Can theorems have exceptions?

Abel was not the first to study this example:

- Euler had discussed it in 1783;
- Lacroix had included it in his *Traité du Calcul Différentiel et du Calcul Intégral* of 1810;
- it had appeared in Fourier's work — which is probably where Abel found it.

Had Cauchy seen it?

Abel: Cauchy's argument is essentially correct, only failing in certain anomalous situations.

Cauchy: no (immediate) reaction — so deemed the example irrelevant?

(See: Henrik Kragh Sørensen, Exceptions and counterexamples: Understanding Abel's comment on *Cauchy's Theorem*, *Historia Mathematica* **32** (2005) 453–480)

Abel's interpretation

Abel read Cauchy as: if the series is convergent at a point x_0 and the individual terms of the series are continuous on a neighbourhood of x_0, then the series is also continuous on that neighbourhood.

But what did Cauchy mean by 'continuity' and 'convergence'?

Note that he only ever defined continuity on an interval — so was it in fact uniform continuity? If, in addition, we regard his notion of convergence on an interval as being uniform convergence, then the theorem holds in all cases.

Similarly, what did Abel mean by 'continuity' and 'convergence'? The same as Cauchy? Or did he use a similar form of words but with a different meaning?

Conditions for Cauchy's theorem to work

Dirichlet (1829): Fourier series can represent discontinuous functions.

But the reason why Cauchy's theorem fails (if indeed it does) remained unclear.

The need for uniform convergence was gradually recognised:

- Karl Weierstrass (lectures in Berlin), 1841;
- Emmanuel Björling (Uppsala), 1846 (or not?);
- Gabriel Stokes (Cambridge), 1847;
- Phillip Seidel (Berlin), 1848.

See: G. H. Hardy, 'Sir George Stokes and the concept of uniform convergence', *Proc. Camb. Phil. Soc.* 19 (1918) 148–156 (also: *Collected Papers of G. H. Hardy*, vol. VII, 505–513)

And: Klaus Viertel, 'The development of the concept of uniform convergence in Karl Weierstrass's lectures and publications between 1861 and 1886', *Arch. Hist. Exact Sci.* 75 (2021), 455–490

Cauchy revisits his theorem (1853)

> *it is easy to see how one can modify the statement of the theorem so that it will no longer have any exception. This is what I am going to explain in a few words.*

Theorem. If the different terms of the series

$$u_0, u_1, u_2, \ldots, u_n, u_{n+1}, \ldots$$

are functions of a real variable x, continuous with respect to this variable within the given limits; and if, in addition, the sum

$$u_n + u_{n+1} + \cdots + u_{n'}$$

always becomes infinitely small for infinitely large values of the whole numbers n and $n' > n$, then the series will be convergent and the sum of the series will be, within the given limits, a continuous function of the variable x.

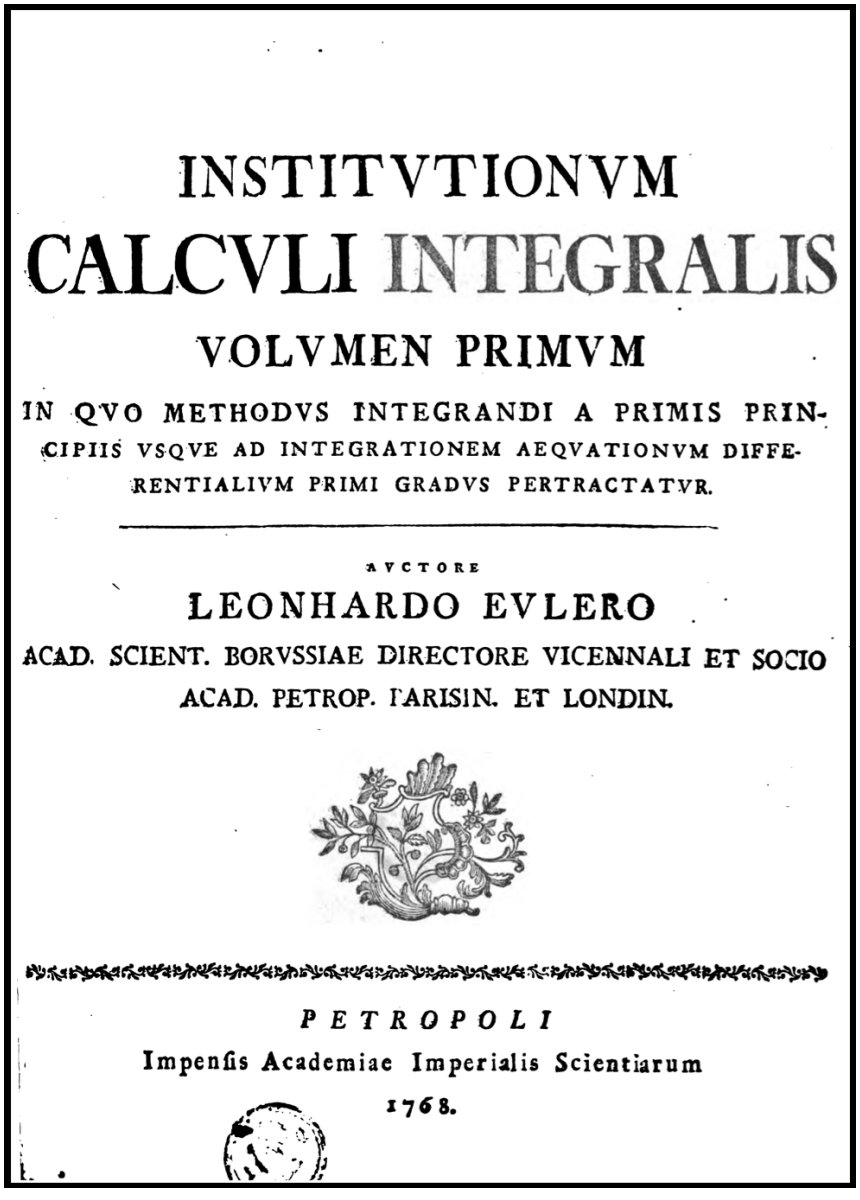
But it was becoming clear that the language of infinities and infinitesimals was inadequate for expressing the problems at hand.

Integration

- Recall that in the 17th century, ‘integration’ was designed for ‘quadrature’, for measuring space or calculating area.

- In the 18th century, ‘integration’ was essentially regarded as the inverse of differentiation.

Integration in the 18th century (1)



INSTITVTIONVM
CALCVLI INTEGRALIS
VOLVMEN PRIMVM
IN QVO METHODVS INTEGRANDI A PRIMIS PRINCIPIIS VSQVE AD INTEGRATIONEM AEQVATIONVM DIFFERENTIALIVM PRIMI GRADVS PERTRACTATVR.

AVCTORE
LEONHARDO EVLERO
ACAD. SCIENT. BORVSSIAE DIRECTORE VICENNALI ET SOCIO ACAD. PETROP. PARISIN. ET LONDIN.

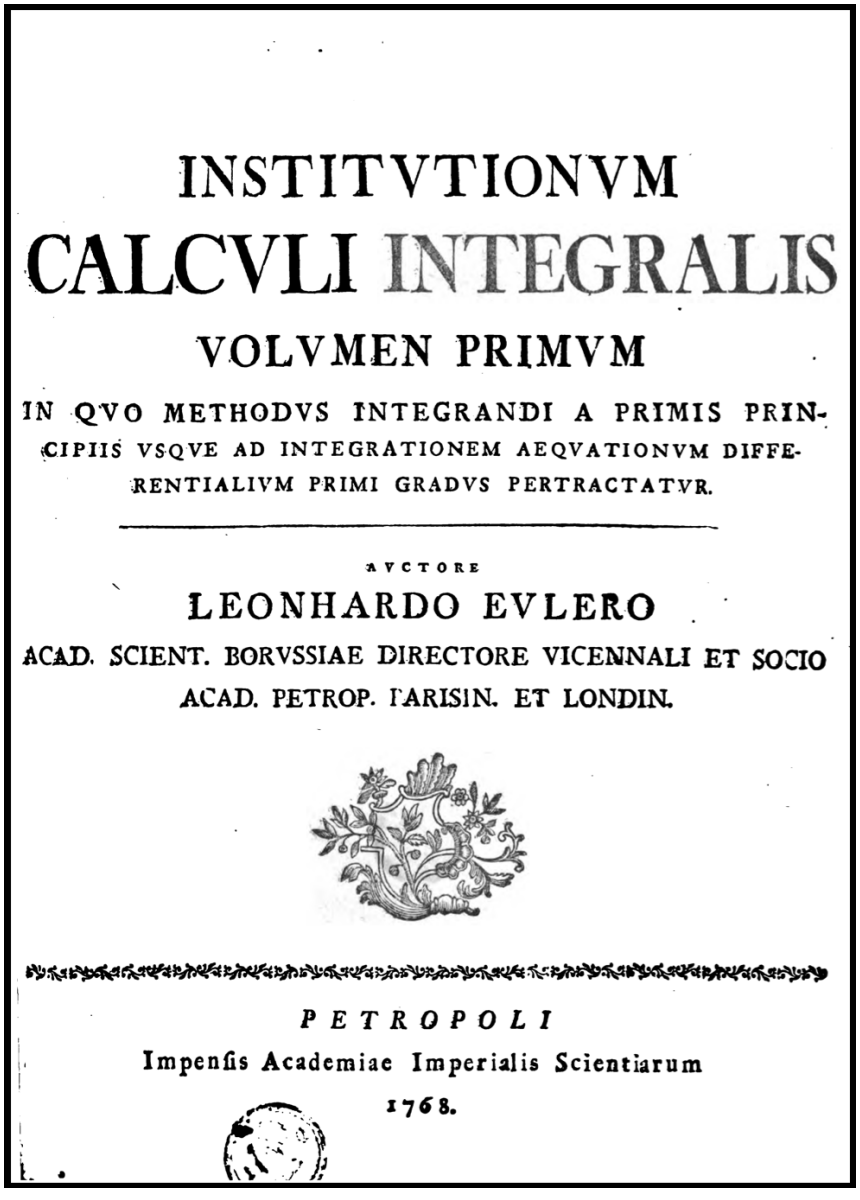
PETROPOLI
Impensis Academiae Imperialis Scientiarum
1768.

Leonhard Euler, *Foundations of integral calculus* (1768):

Definition 1: Integral calculus is the method of finding, from a given relationship between differentials, a relationship between the quantities themselves: and the operation by which this is carried out is usually called integration.

(See *Mathematics emerging*, §14.2.1.)

Integration in the 18th century (2)



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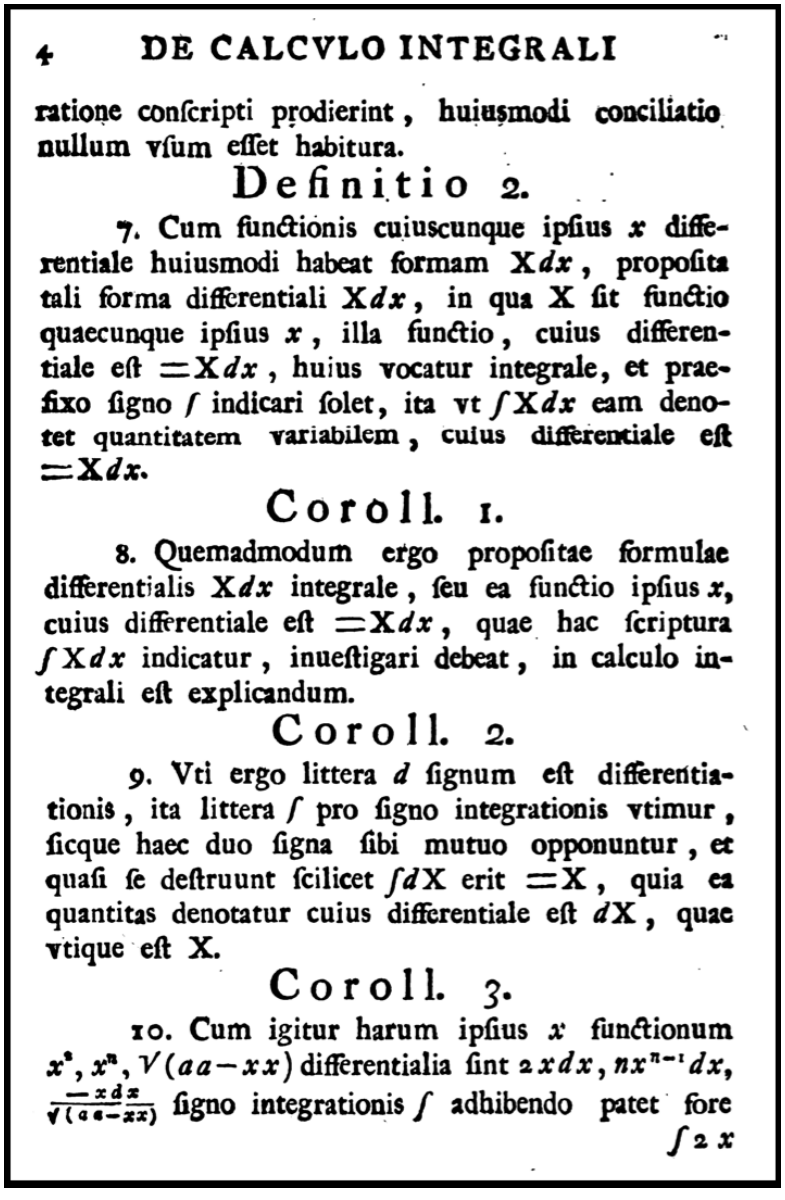
Corollary 1: Therefore where differential calculus teaches us to investigate the relationship between differentials from a given relationship between variable quantities, integral calculus supplies the inverse method.

Corollary 2: Clearly just as in Analysis two operations are always contrary to each other, as subtraction to addition, division to multiplication, extraction of roots to raising of powers, so also by similar reasoning integral calculus is contrary to differential calculus.

Integration in the 18th century (3)



4 DE CALCVLO INTEGRALI

ratione conſcripti prodierint, huiusmodi conciliatio nullum vſum eſſet habitura.

Definitio 2.

7. Cum functionis cuiuscunque ipſius x differentiale huiusmodi habeat formam Xdx, propoſita tali forma differentiali Xdx, in qua X ſit functio quaecunque ipſius x, illa functio, cuius differentiale eſt $=Xdx$, huius vocatur integrale, et praefixo ſigno $\int$ indicari ſolet, ita vt $\int Xdx$ eam denotet quantitatem variabilem, cuius differentiale eſt $=Xdx$.

Coroll. 1.

8. Quemadmodum ergo propoſitae formulae differentialis Xdx integrale, ſeu ea functio ipſius x, cuius differentiale eſt $=Xdx$, quae hac ſcriptura $\int Xdx$ indicatur, inueſtigari debeat, in calculo integrali eſt explicandum.

Coroll. 2.

9. Vti ergo littera d ſignum eſt differentiationis, ita littera $\int$ pro ſigno integrationis vtimur, ſicque haec duo ſigna ſibi mutuo opponuntur, et quaſi ſe deſtruunt ſcilicet $\int dX$ erit $=X$, quia ea quantitas denotatur cuius differentiale eſt dX, quae vtique eſt X.

Coroll. 3.

10. Cum igitur harum ipſius x functionum x^α, x^n, $\sqrt{(aa-xx)}$ differentialia ſint $\alpha x^{\alpha-1}dx$, $nx^{n-1}dx$, $\frac{-xdx}{\sqrt{(aa-xx)}}$ ſigno integrationis $\int$ adhibendo patet fore $\int \alpha x$

Definition 2: Since the differentiation of any function of x has a form of this kind: $X\,dx$, when such a differential form $X\,dx$ is proposed, in which X is any function of x, that function whose differential $= X\,dx$ is called its integral, and is usually indicated by the prefix $\int$, so that $\int X\,dx$ denotes that variable quantity whose differential $= X\,dx$.

Corollary 2: Therefore just as the letter d is the sign of differentiation, so we use the letter $\int$ as the sign of integration, and thus these two signs are mutually contrary to each other, as though they destroy each other: certainly $\int dX = X$, ...

Integration in the 18th century (4)

Coroll. 3.

10. Cum igitur harum ipsius x functionum $x^2, x^n, \sqrt{(aa-xx)}$ differentialia sint $2xdx, nx^{n-1}dx, \frac{-xdx}{\sqrt{(aa-xx)}}$ signo integrationis $\int$ adhibendo patet fore

$$\int 2xdx = xx; \int nx^{n-1}dx = x^n; \int \frac{-xdx}{\sqrt{(aa-xx)}} = \sqrt{(aa-xx)}$$

unde usus huius signi clarius perspicitur.

Some 19th-century ideas

Recall that Fourier coefficients are given by $\frac{2}{\pi}\int_0^\pi \phi(x)\sin nx\,dx$.

It is not always possible to solve such an integral algebraically.

Fourier (1822): but we can draw the curve of $\phi(x)$, and hence that of $\phi(x)\sin nx$, under which there is clearly an area.

Fourier thus returned to the idea of integral as area and influenced Cauchy almost immediately...

A theory of definite integrals (1823)

Cauchy's *Résumé*, 1823, Lesson 21:

Suppose $f(x)$ continuous between $x = x_0$ and $x = X$. Choose $x_1, x_2, \ldots, x_{n-1}$ between these limits. Define

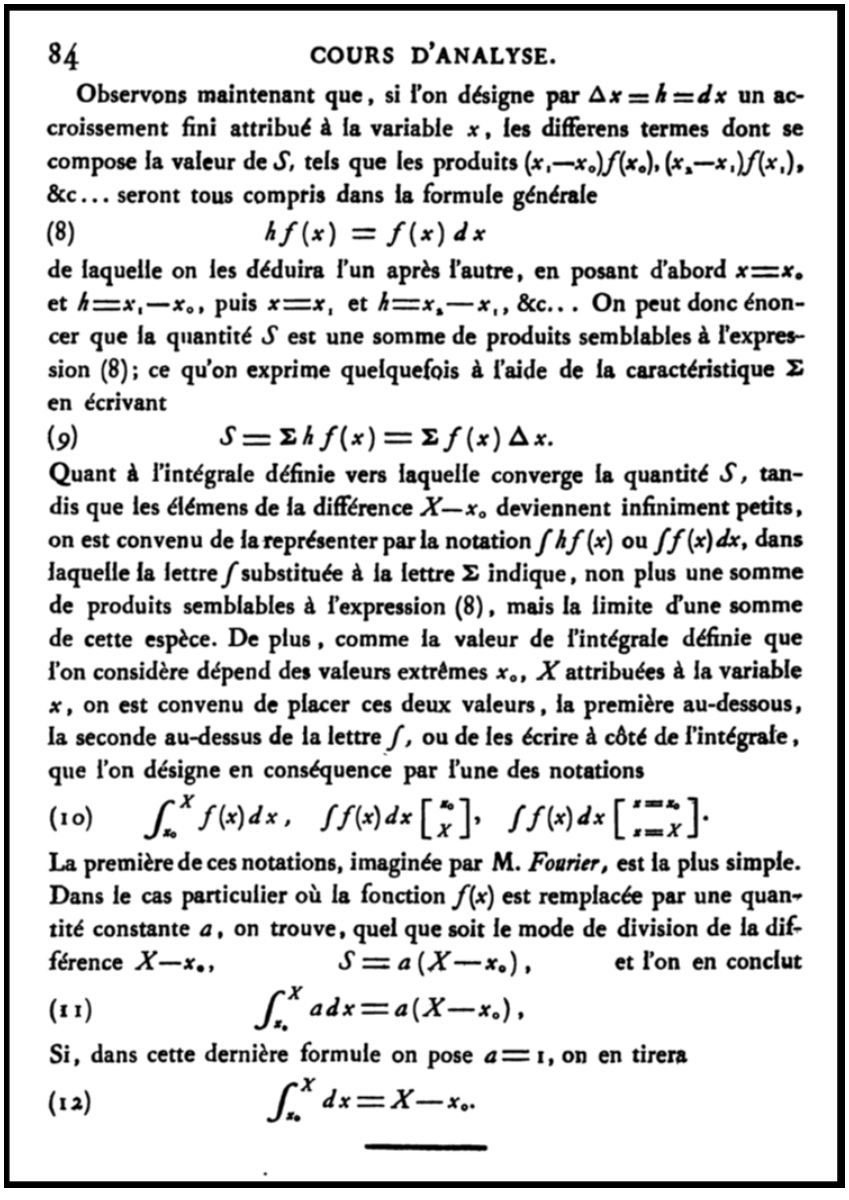
$$S = (x_1 - x_0)f(x_0) + (x_2 - x_1)f(x_1) + \cdots + (X - x_{n-1})f(x_{n-1})$$

[much discussion of dependence on partition followed by]

If the numerical values of the elements are made to decrease indefinitely by increasing their number, the value of S will become essentially constant, or in other words, it will finish by attaining a certain limit which will depend only on the form of the function $f(x)$ and the boundary values $x = x_0$, $x = X$ given to the variable x. This limit is what one calls a definite integral.

[further issues connected with uniform convergence]

Cauchy and integrals



84 COURS D'ANALYSE.

Observons maintenant que, si l'on désigne par $\Delta x = h = dx$ un accroissement fini attribué à la variable x, les differens termes dont se compose la valeur de S, tels que les produits $(x_1 - x_0) f(x_0)$, $(x_2 - x_1) f(x_1)$, &c... seront tous compris dans la formule générale

$$hf(x) = f(x)\,dx \tag{8}$$

de laquelle on les déduira l'un après l'autre, en posant d'abord $x = x_0$ et $h = x_1 - x_0$, puis $x = x_1$ et $h = x_2 - x_1$, &c... On peut donc énoncer que la quantité S est une somme de produits semblables à l'expression (8); ce qu'on exprime quelquefois à l'aide de la caractéristique Σ en écrivant

$$S = \Sigma h f(x) = \Sigma f(x)\,\Delta x. \tag{9}$$

Quant à l'intégrale définie vers laquelle converge la quantité S, tandis que les élémens de la différence $X - x_0$ deviennent infiniment petits, on est convenu de la représenter par la notation $\int h f(x)$ ou $\int f(x)dx$, dans laquelle la lettre $\int$ substituée à la lettre Σ indique, non plus une somme de produits semblables à l'expression (8), mais la limite d'une somme de cette espèce. De plus, comme la valeur de l'intégrale définie que l'on considère dépend des valeurs extrêmes x_0, X attribuées à la variable x, on est convenu de placer ces deux valeurs, la première au-dessous, la seconde au-dessus de la lettre $\int$, ou de les écrire à côté de l'intégrale, que l'on désigne en conséquence par l'une des notations

$$\int_{x_0}^{X} f(x)\,dx, \quad \int f(x)\,dx \begin{bmatrix} x_0 \\ X \end{bmatrix}, \quad \int f(x)\,dx \begin{bmatrix} x = x_0 \\ x = X \end{bmatrix}. \tag{10}$$

La première de ces notations, imaginée par M. *Fourier*, est la plus simple. Dans le cas particulier où la fonction $f(x)$ est remplacée par une quantité constante a, on trouve, quel que soit le mode de division de la différence $X - x_0$, $S = a(X - x_0)$, et l'on en conclut

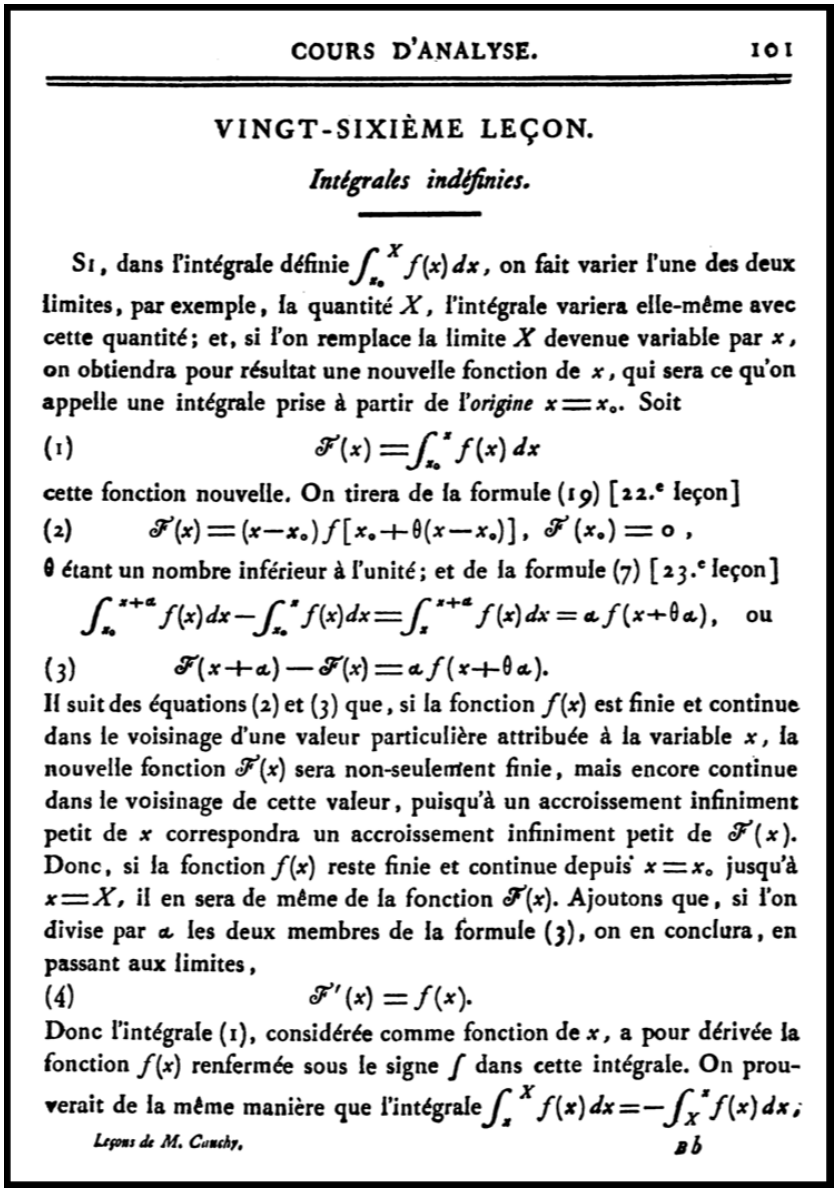
$$\int_{x_0}^{X} a\,dx = a(X - x_0), \tag{11}$$

Si, dans cette dernière formule on pose $a = 1$, on en tirera

$$\int_{x_0}^{X} dx = X - x_0. \tag{12}$$

Is it valid to use the symbol $\int$ here?

Cauchy and the Fundamental Theorem of Calculus



COURS D'ANALYSE. 101

VINGT-SIXIÈME LEÇON.

Intégrales indéfinies.

Si, dans l'intégrale définie $\int_{x_0}^{X} f(x)\,dx$, on fait varier l'une des deux limites, par exemple, la quantité X, l'intégrale variera elle-même avec cette quantité; et, si l'on remplace la limite X devenue variable par x, on obtiendra pour résultat une nouvelle fonction de x, qui sera ce qu'on appelle une intégrale prise à partir de l'*origine* $x = x_0$. Soit

$$(1) \qquad \mathcal{F}(x) = \int_{x_0}^{x} f(x)\,dx$$

cette fonction nouvelle. On tirera de la formule (19) [22.e leçon]

$$(2) \qquad \mathcal{F}(x) = (x - x_0) f[x_0 + \theta(x - x_0)], \quad \mathcal{F}(x_0) = 0,$$

θ étant un nombre inférieur à l'unité; et de la formule (7) [23.e leçon]

$$\int_{x_0}^{x+\alpha} f(x)dx - \int_{x_0}^{x} f(x)dx = \int_{x}^{x+\alpha} f(x)dx = \alpha f(x + \theta\alpha), \quad \text{ou}$$

$$(3) \qquad \mathcal{F}(x + \alpha) - \mathcal{F}(x) = \alpha f(x + \theta\alpha).$$

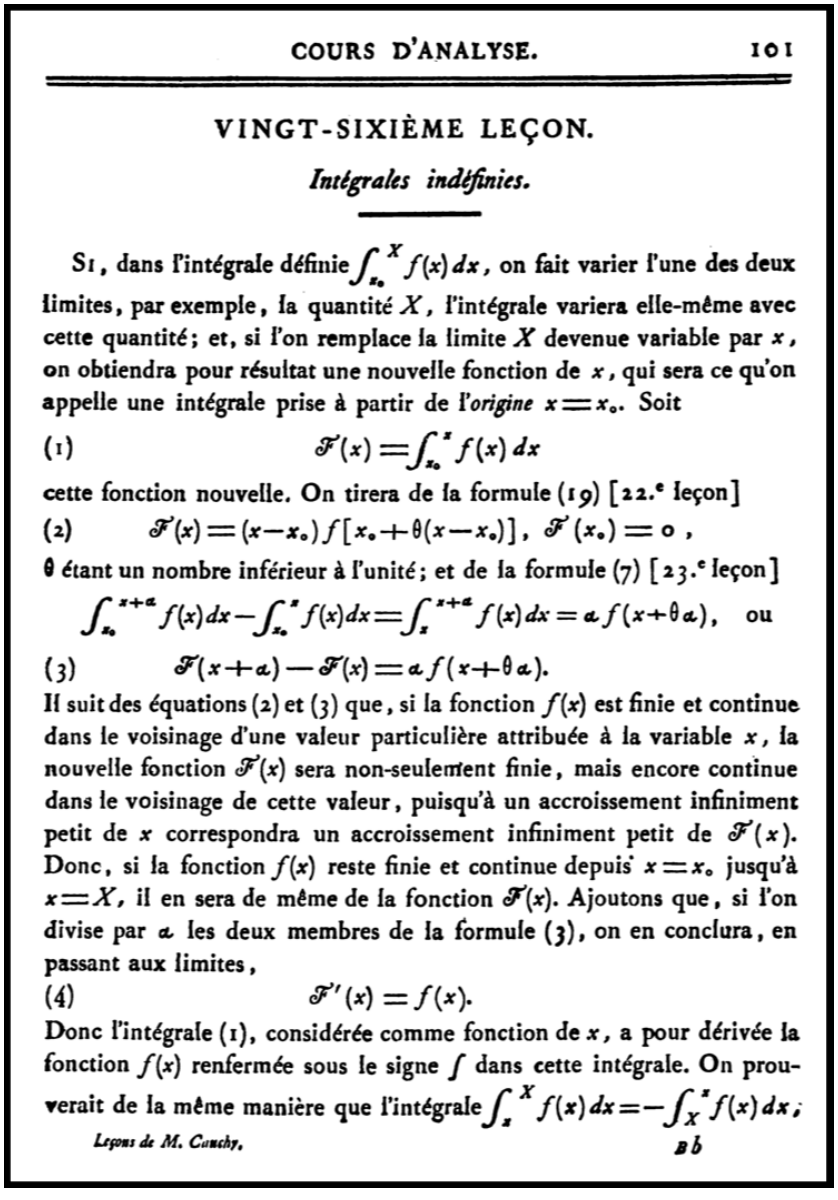
Il suit des équations (2) et (3) que, si la fonction $f(x)$ est finie et continue dans le voisinage d'une valeur particulière attribuée à la variable x, la nouvelle fonction $\mathcal{F}(x)$ sera non-seulement finie, mais encore continue dans le voisinage de cette valeur, puisqu'à un accroissement infiniment petit de x correspondra un accroissement infiniment petit de $\mathcal{F}(x)$. Donc, si la fonction $f(x)$ reste finie et continue depuis $x = x_0$ jusqu'à $x = X$, il en sera de même de la fonction $\mathcal{F}(x)$. Ajoutons que, si l'on divise par α les deux membres de la formule (3), on en conclura, en passant aux limites,

$$(4) \qquad \mathcal{F}'(x) = f(x).$$

Donc l'intégrale (1), considérée comme fonction de x, a pour dérivée la fonction $f(x)$ renfermée sous le signe $\int$ dans cette intégrale. On prouverait de la même manière que l'intégrale $\int_{x}^{X} f(x)\,dx = -\int_{X}^{x} f(x)\,dx$;

If in the definite integral $\int_{x_0}^{X} f(x)\,dx$ *one makes one of the two limits vary, for example the quantity* X*, the integral itself will vary with this quantity; and if one replaces the variable limit* X *by* x*, there results a new function of* x*, . . .*

Cauchy and the Fundamental Theorem of Calculus



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Leçons de M. Cauchy. Bb

Let

$$\mathscr{F}(x) = \int_{x_0}^{x} f(x)\,dx$$

be this new function.

Proved that $\mathscr{F}'(x) = f(x)$, and also that

$$\varpi'(x) = 0 \Rightarrow \varpi(x) = \text{const},$$

which may be used to show that if $F'(x) = f(x)$, then

$$\int_{x_0}^{X} f(x)\,dx = F(X) - F(x_0).$$

The Fundamental Theorem of Calculus

What is the Fundamental Theorem of Calculus?

- integration is the inverse of differentiation?
- integration 'as a sum' is the same as integration 'by rule'?
- Cauchy's integration is the same as Euler's integration?
- 19th-century integration is the same as 18th-century integration?
- ...

Bernhard Riemann (1826–1866)

Ueber die

Darstellbarkeit einer Function

durch eine trigonometrische Reihe.

Von

B. Riemann.

Aus dem dreizehnten Bande der Abhandlungen der Königlichen Gesellschaft der Wissenschaften zu Göttingen.

Göttingen,
in der Dieterichschen Buchhandlung.
1867.

Riemann's integral (1853)

Function $f(x)$ no longer required to be continuous on $[a, b]$. Take $x_1 < x_2 < \cdots < x_{n-1}$. Define $\delta_1 := x_1 - a$, $\delta_2 := x_2 - x_1$, ..., $\delta_n := b - x_{n-1}$. Choose numbers ε_i between 0 and 1. Then define

$$S := \delta_1 f(a + \varepsilon_1 \delta_1) + \delta_2 f(x_1 + \varepsilon_2 \delta_2) + \delta_3 f(x_2 + \varepsilon_3 \delta_3) + \cdots + \delta_n f(a_{n-1} + \varepsilon_n \delta_n)$$

If this has the property that it comes infinitely close to a fixed value A when all the δ_i become infinitely small, then this is the value of $\int_a^b f(x)\,dx$.

Many variants over the years, all called Riemann integral.

Lebesgue's integral (1901)

Considers step functions on subsets that are not necessarily intervals, thus requiring the notion of a measure (Borel, 1894).

Results in a notion of integral of wider applicability than Riemann's; for example:

can integrate highly discontinuous functions, such as the Dirichlet function:

$$f(x) := \begin{cases} 1 & \text{if } x \text{ is rational;} \\ 0 & \text{if } x \text{ is irrational.} \end{cases}$$

BO1 History of Mathematics
Lecture XII
19th-century rigour in real analysis, part 2: real numbers and sets

MT 2022 Week 6

Summary

- Proofs of the Intermediate Value Theorem revisited
- Convergence and completeness
- Dedekind and the continuum
- Cantor and numbers and sets
- Where and when did sets emerge?
- Early set theory
- Set theory as a language

The Intermediate Value Theorem (1)

Bolzano's criticisms (1817) of existing proofs:

> *The most common kind of proof depends on a truth borrowed from geometry . . . But it is clear that it is an intolerable offense against correct method to derive truths of pure (or general) mathematics (i.e., arithmetic, algebra, analysis) from considerations which belong to a merely applied (or special) part, namely, geometry.*

His own proof includes something close to a proof that Cauchy sequences converge:

> *. . . the true value of X* [*the limit*] *therefore ... can be determined as accurately as required . . . There is, therefore, a real quantity which the terms of the series, if it is continued far enough, approach as closely as desired.*

But Bolzano assumed the existence of the limit.

The Intermediate Value Theorem (2)

Cauchy's 1st proof (*Cours d'analyse*, 1821, p. 44) is geometric (though he didn't provide a diagram):

> *The function $f(x)$ being continuous between the limits $x = x_0$, $x = X$, the curve which has for equation $y = f(x)$ passes first through the point corresponding to the coordinates x_0, $f(x_0)$, second through the point corresponding to the coordinates X, $f(X)$, will be continuous between these two points: and, since the constant ordinate b of the line which has for equation $y = b$ is to be found between the ordinates $f(x_0)$, $f(X)$ of the two points under consideration, the line will necessarily pass between these two points, which it cannot do without meeting the curve mentioned above in the interval.*

Cauchy's 2nd proof in a different context (p. 460): a numerical method for finding roots of equations — tacitly assumes that bounded monotone sequences of real numbers converge [see Lecture VIII].

The need for a deeper understanding (1)

Emergence of rigour in Analysis:

- Bolzano, *Rein analytischer Beweis ...*, 1817;
- Cauchy, *Cours d'analyse*, 1821, etc.

By 1821, therefore, attempts to prove the intermediate value theorem had brought three important propositions into play:

1. *Cauchy sequences are convergent (with an unsuccessful proof by Bolzano in 1817; accepted without proof by Cauchy in 1821).*
2. *A [non-empty] set of numbers bounded below has a greatest lower bound (proved by Bolzano in 1817 on the basis of (1)).*
3. *A monotonic bounded sequence converges to a limit (taken for granted by Cauchy in 1821).*

(*Mathematics emerging*, §16.3.1.)

The need for a deeper understanding (2)

What Bolzano and Cauchy missed: completeness

Completeness of the real number system $\mathbb{R}$ in modern teaching:

- non-empty bounded sets of real numbers have least upper bounds
- monotonic bounded sequences converge
- Cauchy sequences converge
- ...

All equivalent

Equivalence of formulations of completeness

Bolzano–Weierstrass Theorem: A bounded sequence of real numbers has a convergent subsequence.

Implicit in Bolzano (1817); explicit in lectures by Karl Weierstrass (1815–1897) in Berlin 1859/60, 1863/64: a step in proofs from other definitions of completeness that Cauchy sequences converge.

Modern proofs often use the lemma that every infinite sequence of real numbers has an infinite monotonic subsequence.

How to incorporate these ideas into analysis in a rigorous way?

All of the above relies upon an intuitive notion of real number — so perhaps provide a formal definition of these? One that includes the idea of completeness?

Richard Dedekind (1831–1916)

Stetigkeit

und

irrationale Zahlen.

102194

Von

Richard Dedekind,
Professor der höheren Mathematik am Collegium Carolinum zu Braunschweig.

Braunschweig,
Druck und Verlag von Friedrich Vieweg und Sohn.
1872.

Dedekind and the foundations of analysis

Teaching calculus in the Zürich Polytechnic (1858), later (from 1862) teaching Fourier series in the Braunschweig Polytechnic, found himself dissatisfied with:

- geometry as a foundation for analysis;
- tacit assumptions about convergence (for monotonic bounded sequences, for example).

Response eventually published in *Stetigkeit und irrationale Zahlen* (1872) [translated as *Continuity and irrational numbers* by Wooster Woodruff Beman, 1901]

Dedekind and continuity (1)

Intuition suggests that numbers (an arithmetical concept) should have the same completeness and continuity properties as a line (a geometrical concept). But we must define these concepts for numbers without appeal to geometrical intuition.

Geometrically, every point separates a line into two parts.

> *I find the essence of continuity in the converse, i.e., in the following principle:*
>
> *"If all points of the straight line fall into two classes such that every point of the first class lies to the left of every point of the second class, then there exists one and only one point which produces this division of all points into two classes, this severing of the straight line into two portions."*

Dedekind and continuity (2)

But Dedekind couldn't *prove* this property, so he had to take it as an axiom:

> *The assumption of this property for the line is nothing but an Axiom, through which alone we attribute continuity to the line, through which we understand continuity in the line.*

(See *Mathematics emerging*, §16.3.2.)

Dedekind and continuity (3)

Next adapt this idea to the arithmetical context:

- every number x separates all other numbers into two classes — those greater than x, and those less than x;
- conversely, every such separation of numbers defines a number.

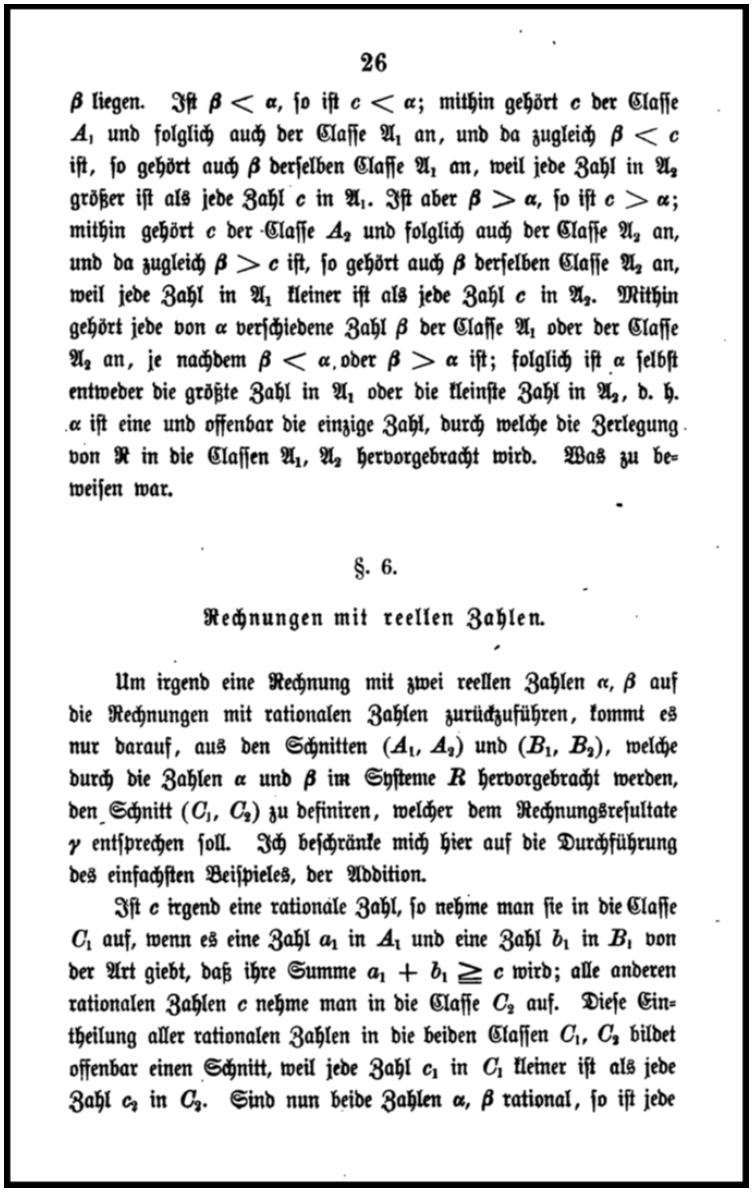
Hence Dedekind cuts (or sections, from the original German Schnitt).

Dedekind cuts (1)

- Start from the system of rational numbers R (assumed known)
- Separate R into two classes A_1 and A_2 such that
  - for any a_1 in A_1, $a_1 < a_2$ for every a_2 in A_2
  - for any a_2 in A_2, $a_2 > a_1$ for every a_1 in A_1
- The cut denoted by (A_1, A_2) defines a number
- Important observation: (A_1, A_2) need not be rational

Whenever, then, we have to do with a cut produced by no rational number, we create a new irrational number, which we regard as completely defined by this cut ...

Dedekind cuts (2)



26

β liegen. Ist $\beta < \alpha$, so ist $c < \alpha$; mithin gehört c der Classe A_1 und folglich auch der Classe $\mathfrak{A}_1$ an, und da zugleich $\beta < c$ ist, so gehört auch β derselben Classe $\mathfrak{A}_1$ an, weil jede Zahl in $\mathfrak{A}_2$ größer ist als jede Zahl c in $\mathfrak{A}_1$. Ist aber $\beta > \alpha$, so ist $c > \alpha$; mithin gehört c der Classe A_2 und folglich auch der Classe $\mathfrak{A}_2$ an, und da zugleich $\beta > c$ ist, so gehört auch β derselben Classe $\mathfrak{A}_2$ an, weil jede Zahl in $\mathfrak{A}_1$ kleiner ist als jede Zahl c in $\mathfrak{A}_2$. Mithin gehört jede von α verschiedene Zahl β der Classe $\mathfrak{A}_1$ oder der Classe $\mathfrak{A}_2$ an, je nachdem $\beta < \alpha$ oder $\beta > \alpha$ ist; folglich ist α selbst entweder die größte Zahl in $\mathfrak{A}_1$ oder die kleinste Zahl in $\mathfrak{A}_2$, d. h. α ist eine und offenbar die einzige Zahl, durch welche die Zerlegung von $\mathfrak{R}$ in die Classen $\mathfrak{A}_1$, $\mathfrak{A}_2$ hervorgebracht wird. Was zu beweisen war.

§. 6.

Rechnungen mit reellen Zahlen.

Um irgend eine Rechnung mit zwei reellen Zahlen α, β auf die Rechnungen mit rationalen Zahlen zurückzuführen, kommt es nur darauf, aus den Schnitten (A_1, A_2) und (B_1, B_2), welche durch die Zahlen α und β im Systeme R hervorgebracht werden, den Schnitt (C_1, C_2) zu definiren, welcher dem Rechnungsresultate γ entsprechen soll. Ich beschränke mich hier auf die Durchführung des einfachsten Beispieles, der Addition.

Ist c irgend eine rationale Zahl, so nehme man sie in die Classe C_1 auf, wenn es eine Zahl a_1 in A_1 und eine Zahl b_1 in B_1 von der Art giebt, daß ihre Summe $a_1 + b_1 \geqq c$ wird; alle anderen rationalen Zahlen c nehme man in die Classe C_2 auf. Diese Eintheilung aller rationalen Zahlen in die beiden Classen C_1, C_2 bildet offenbar einen Schnitt, weil jede Zahl c_1 in C_1 kleiner ist als jede Zahl c_2 in C_2. Sind nun beide Zahlen α, β rational, so ist jede

Dedekind showed how to add two cuts, and how to use them in limiting arguments — but did little else with them.

Significance: a major step towards

- understanding completeness, and
- giving a rigorous definition of an irrational number, hence
- setting the foundations of analysis onto a sound logical basis.

Dissemination of Dedekind's ideas

Stetigkeit und irrationale Zahlen reprinted many times, often in conjunction with the later essay *Was sind und was sollen die Zahlen?* (1888) [see below].

Translated into English as *Essays on the theory of numbers* by Wooster Woodruff Beman (1901).

Popularised and organised for teaching, starting from Peano axioms for natural numbers, by Edmund Landau in *Grundlagen der Analysis* [*Foundations of analysis*] (1930), a book that contains very few words.

A good modern (historically sensitive) account can be found in: Leo Corry, *A brief history of numbers*, OUP, 2015, §10.6.

Other approaches

Georg Cantor (1872) and Eduard Heine (1872) created real numbers as equivalence classes of Cauchy sequences of rational numbers. (Also: Charles Méray in 1869.)

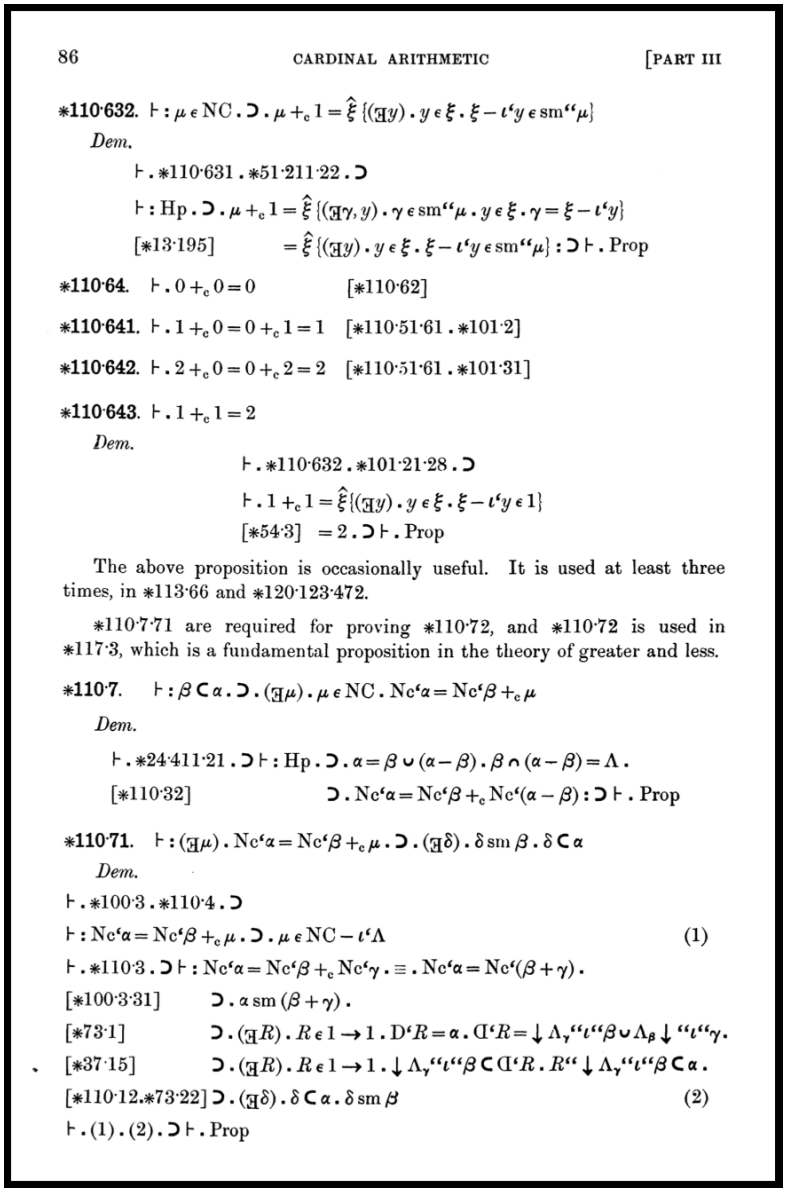
(On Cantor's approach, see *Mathematics emerging*, §16.3.3.)

Heine acknowledged a debt to Cantor and a debt to the lectures of Weierstrass.

Later constructions by many mathematicians and philosophers, often as part of a broader effort to lay down logical foundations for mathematics as a whole — for example:

- Carl Johannes Thomae, 1880, 1890
- Giuseppe Peano, 1889, 1891
- Gottlob Frege, 1884, 1893, 1903
- Otto Hölder, 1901

Extreme formalism



86 CARDINAL ARITHMETIC [PART III

$*110{\cdot}632.\ \vdash : \mu \in \mathrm{NC} . \supset . \mu +_c 1 = \hat{\xi}\{(\exists y) . y \in \xi . \xi - \iota`y \in \mathrm{sm}``\mu\}$

Dem.

$\vdash . *110{\cdot}631 . *51{\cdot}211{\cdot}22 . \supset$

$\vdash : \mathrm{Hp} . \supset . \mu +_c 1 = \hat{\xi}\{(\exists \gamma, y) . \gamma \in \mathrm{sm}``\mu . y \in \xi . \gamma = \xi - \iota`y\}$

$[*13{\cdot}195] \quad = \hat{\xi}\{(\exists y) . y \in \xi . \xi - \iota`y \in \mathrm{sm}``\mu\} : \supset \vdash . \mathrm{Prop}$

$*110{\cdot}64.\ \vdash . 0 +_c 0 = 0 \quad [*110{\cdot}62]$

$*110{\cdot}641.\ \vdash . 1 +_c 0 = 0 +_c 1 = 1 \quad [*110{\cdot}51{\cdot}61 . *101{\cdot}2]$

$*110{\cdot}642.\ \vdash . 2 +_c 0 = 0 +_c 2 = 2 \quad [*110{\cdot}51{\cdot}61 . *101{\cdot}31]$

$*110{\cdot}643.\ \vdash . 1 +_c 1 = 2$

Dem.

$\vdash . *110{\cdot}632 . *101{\cdot}21{\cdot}28 . \supset$

$\vdash . 1 +_c 1 = \hat{\xi}\{(\exists y) . y \in \xi . \xi - \iota`y \in 1\}$

$[*54{\cdot}3] \quad = 2 . \supset \vdash . \mathrm{Prop}$

The above proposition is occasionally useful. It is used at least three times, in *113·66 and *120·123·472.

*110·7·71 are required for proving *110·72, and *110·72 is used in *117·3, which is a fundamental proposition in the theory of greater and less.

$*110{\cdot}7.\ \vdash : \beta \subset \alpha . \supset . (\exists \mu) . \mu \in \mathrm{NC} . \mathrm{Nc}`\alpha = \mathrm{Nc}`\beta +_c \mu$

Dem.

$\vdash . *24{\cdot}411{\cdot}21 . \supset \vdash : \mathrm{Hp} . \supset . \alpha = \beta \cup (\alpha - \beta) . \beta \cap (\alpha - \beta) = \Lambda .$

$[*110{\cdot}32] \quad \supset . \mathrm{Nc}`\alpha = \mathrm{Nc}`\beta +_c \mathrm{Nc}`(\alpha - \beta) : \supset \vdash . \mathrm{Prop}$

$*110{\cdot}71.\ \vdash : (\exists \mu) . \mathrm{Nc}`\alpha = \mathrm{Nc}`\beta +_c \mu . \supset . (\exists \delta) . \delta \mathrm{sm} \beta . \delta \subset \alpha$

Dem.

$\vdash . *100{\cdot}3 . *110{\cdot}4 . \supset$

$\vdash : \mathrm{Nc}`\alpha = \mathrm{Nc}`\beta +_c \mu . \supset . \mu \in \mathrm{NC} - \iota`\Lambda \quad (1)$

$\vdash . *110{\cdot}3 . \supset \vdash : \mathrm{Nc}`\alpha = \mathrm{Nc}`\beta +_c \mathrm{Nc}`\gamma . \equiv . \mathrm{Nc}`\alpha = \mathrm{Nc}`(\beta + \gamma) .$

$[*100{\cdot}3{\cdot}31] \quad \supset . \alpha \mathrm{sm} (\beta + \gamma) .$

$[*73{\cdot}1] \quad \supset . (\exists R) . R \in 1 \to 1 . \mathrm{D}`R = \alpha . \mathrm{C}`R = \downarrow \Lambda_\gamma ``\iota ``\beta \cup \Lambda_\beta \downarrow ``\iota ``\gamma .$

$[*37{\cdot}15] \quad \supset . (\exists R) . R \in 1 \to 1 . \downarrow \Lambda_\gamma ``\iota ``\beta \subset \mathrm{C}`R . R`` \downarrow \Lambda_\gamma ``\iota ``\beta \subset \alpha .$

$[*110{\cdot}12.*73{\cdot}22] \supset . (\exists \delta) . \delta \subset \alpha . \delta \mathrm{sm} \beta \quad (2)$

$\vdash . (1) . (2) . \supset \vdash . \mathrm{Prop}$

Alfred North Whitehead and Bertrand Russell, *Principia mathematica*, 3 vols., Cambridge University Press, 1910, 1912, 1913

Vol. II, p. 86: $1 + 1 = 2$

"The above proposition is occasionally useful."

NB. This is not the source of our axioms for the reals.

See: *Logicomix: An epic search for truth* (2009)

New ideas

An idea that emerged as central to Dedekind's work: that of a set

In fact, naive notions of sets had already appeared all but unremarked earlier in the nineteenth century

- as Gauss' classes, orders, genera (of binary quadratic forms with integer coefficients) [see Lecture XIV];
- as Galois' groupes (of permutations and of substitutions);
- as Cauchy's systèmes (of substitutions);
- as Dedekind's Zahlkörpern (of algebraic numbers).

This is by no means an exhaustive list of examples; see *Mathematics emerging*, §18.2 for others.

Formalisation of the concept of a set

Georg Cantor: series of articles in *Mathematische Annalen*, 1879–1883

Final one also published separately as *Grundlagen einer allgemeinen Mannigfaltigkeitslehre* [*Foundations of a general theory of aggregates*], Teubner, Leipzig, 1883:

> *By an "aggregate" (Menge) we are to understand any collection into a whole (Zusammenfassung zu einem Ganzen) M of definite and separate objects m of our intuition or our thought.*

Cantor and the continuum

Cantor's major interest: the continuum (i.e., the set of real numbers).

How to characterise this set within the collection of all sets? — A question that Cantor never satisfactorily answered.

Cantor's first great insight regarding sets (1873): infinite sets can have different sizes.

Cantor's first proof that the continuum is uncountable

Proposition: Given any sequence of real numbers $\omega_1, \omega_2, \omega_3, \ldots$ and any interval $[\alpha, \beta]$, there is a real number in $[\alpha, \beta]$ that is not contained in the given sequence.

Proof proceeds by construction of a sequence of nested intervals $[\alpha, \beta] \supseteq [\alpha_1, \beta_1] \supseteq [\alpha_2, \beta_2] \supseteq [\alpha_3, \beta_3] \supseteq \cdots$. Cantor considered the different cases where the sequence terminates or does not, but in all instances he constructed a real number in the interval that does not lie in the original sequence.

Next suppose that the continuum is countable, i.e., that the real numbers may be listed $\omega_1, \omega_2, \omega_3, \ldots$. But then there is a real number in any interval $[\alpha, \beta]$ that does not belong to this list — a contradiction.

The more famous diagonal argument came later (1891).

One-to-one correspondences

Also in the 1874 paper:

The algebraic $\mathbb{A}$ numbers are countable — therefore transcendental numbers exist.

> **NB:** In 1851 Joseph Liouville had already produced a constructive proof of the existence of transcendental numbers.
>
> Charles Hermite proved in 1873 that e is transcendental.
>
> Proof of the transcendence of π was finally accomplished by Carl Louis Lindemann in 1882.

Cantor to Dedekind (1877): there is a one-to-one correspondence between a line and the plane — "Je le vois, mais je ne le crois pas!" ("I see it, but I don't believe it!")

Cantor's *Mengenlehre*

Developed at the end of the nineteenth century (1878–1897): a general theory of sets and of transfinite numbers — infinite cardinals (e.g., $\#\mathbb{N} = \aleph_0$, $\#\mathbb{R} = c$), transfinite ordinals, ...

Mixed terminology: Inbegriff, System, Mannigfaltigkeit, Menge

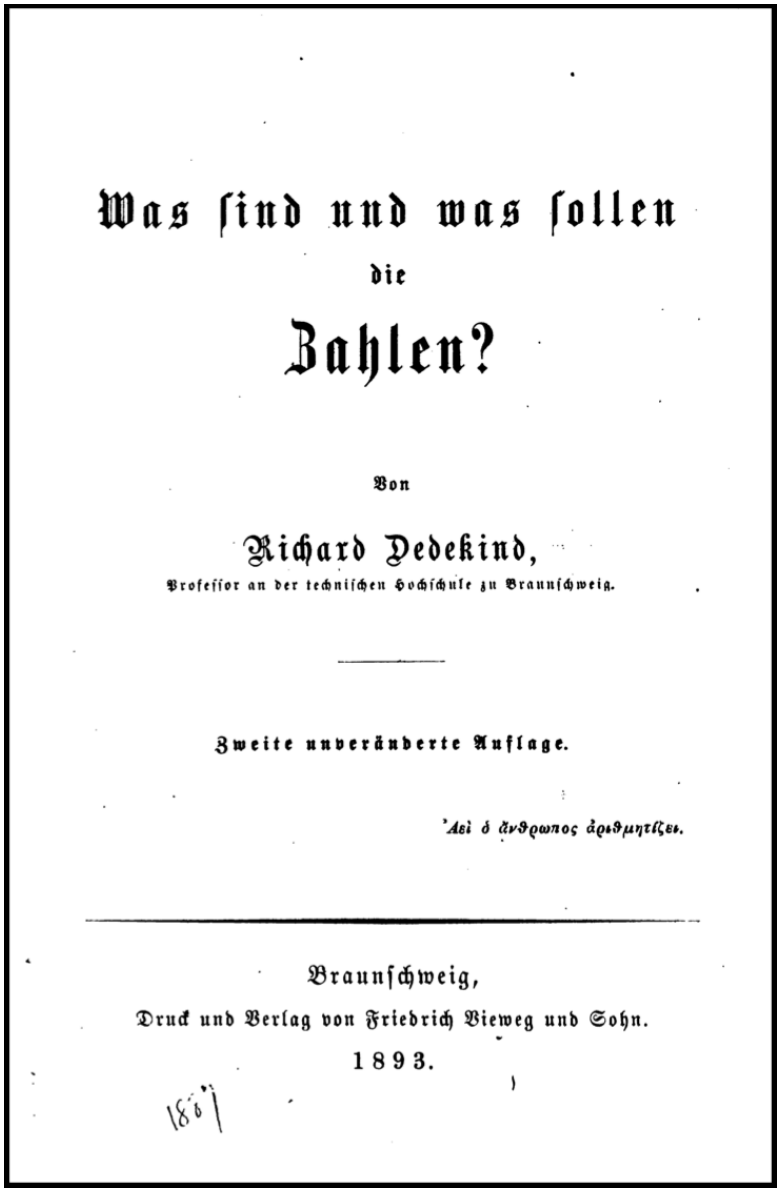
Continuum hypothesis (1878): there is no infinite cardinal strictly between $\aleph_0$ and c

Power set construction given in 1890: $\mathscr{P}(S)$ — the set of all subsets of a set S

Cantor's Theorem: $\#\mathscr{P}(S) > \#S$

Further: $\#\mathscr{P}(\mathbb{N}) = \#\mathbb{R}$, or $2^{\aleph_0} = c$

Was sind und was sollen die Zahlen?



Was ſind und was ſollen
die
Zahlen?

Von

Richard Dedekind,
Profeſſor an der techniſchen Hochſchule zu Braunſchweig.

Zweite unveränderte Auflage.

Ἀεὶ ὁ ἄνθρωπος ἀριθμητίζει.

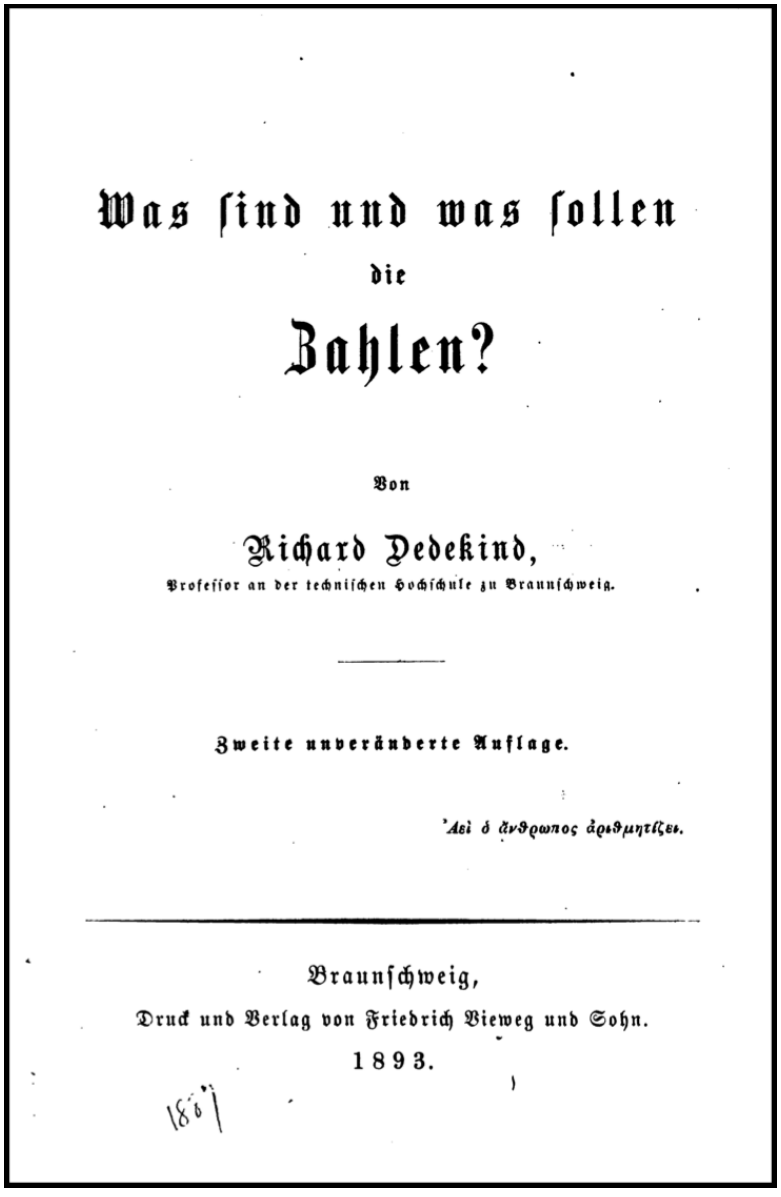
Braunſchweig,
Druck und Verlag von Friedrich Vieweg und Sohn.
1893.

Richard Dedekind, *Was sind und was sollen die Zahlen?* Braunschweig, 1893

Contains, amongst other things:

- a definition of infinite sets;
- an axiomatisation of the natural numbers (soon simplified by Peano).

Was sind und was sollen die Zahlen?



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Braunſchweig,
Druck und Verlag von Friedrich Vieweg und Sohn.
1893.

Also includes a definition of a function as a mapping between sets (p. 6):

"By a mapping of a system S we understand a law according to which every determinate element s of S is associated with a determinate thing which is called the *image* of s and is denoted by $\phi(s)$ ..."

Was sind und was sollen die Zahlen?

Extract from William Ewald, *From Kant to Hilbert: a source book in the foundations of mathematics*, OUP, 1996, vol. II, p. 790:

> *The title of Dedekind's paper is subtle: rigidly translated it asks 'What are, and what ought to be, the numbers?' But sollen here carries several senses—among them, 'What is the best way to regard the numbers?'; 'What is the function of numbers?; 'What are numbers supposed to be?'. But perhaps Dedekind's title is famous enough to be left in the original.*

W. W. Beman translated the essay under the title *The nature and meaning of numbers* (1901).

Was sind und was sollen die Zahlen?

(wegen der Aehnlichkeit von φ) auch a' und jedes Element u' verſchieden von a und folglich in T enthalten ſein; mithin iſt $\psi(T) \subset T$, und da T endlich iſt, ſo muß $\psi(T) = T$, alſo $\mathfrak{M}(a', U') = T$ ſein. Hieraus folgt aber (nach 15)

$$\mathfrak{M}(a', a, U') = \mathfrak{M}(a, T),$$

d. h. nach dem Obigen $S' = S$. Alſo iſt auch in dieſem Falle der erforderliche Beweis geführt.

§. 6.

Einfach unendliche Syſteme. Reihe der natürlichen Zahlen.

71. Erklärung. Ein Syſtem N heißt einfach unendlich, wenn es eine ſolche ähnliche Abbildung φ von N in ſich ſelbſt giebt, daß N als Kette (44) eines Elementes erſcheint, welches nicht in $\varphi(N)$ enthalten iſt. Wir nennen dies Element, das wir im Folgenden durch das Symbol 1 bezeichnen wollen, das Grundelement von N und ſagen zugleich, das einfach unendliche Syſtem N ſei durch dieſe Abbildung φ geordnet. Behalten wir die früheren bequemen Bezeichnungen für die Bilder und Ketten bei (§. 4), ſo beſteht mithin das Weſen eines einfach unendlichen Syſtems N in der Exiſtenz einer Abbildung φ von N und eines Elementes 1, die den folgenden Bedingungen α, β, γ, δ genügen:

α. $N' \subset N$.

β. $N = 1_0$.

γ. Das Element 1 iſt nicht in N' enthalten.

δ. Die Abbildung φ iſt ähnlich.

Offenbar folgt aus α, γ, δ, daß jedes einfach unendliche Syſtem N wirklich ein unendliches Syſtem iſt (64), weil es einem echten Theile N' ſeiner ſelbſt ähnlich iſt.

72. Satz. In jedem unendlichen Syſteme S iſt ein einfach unendliches Syſtem N als Theil enthalten.

Written in an explicitly set-theoretic language

(But with slightly different notation from ours.)

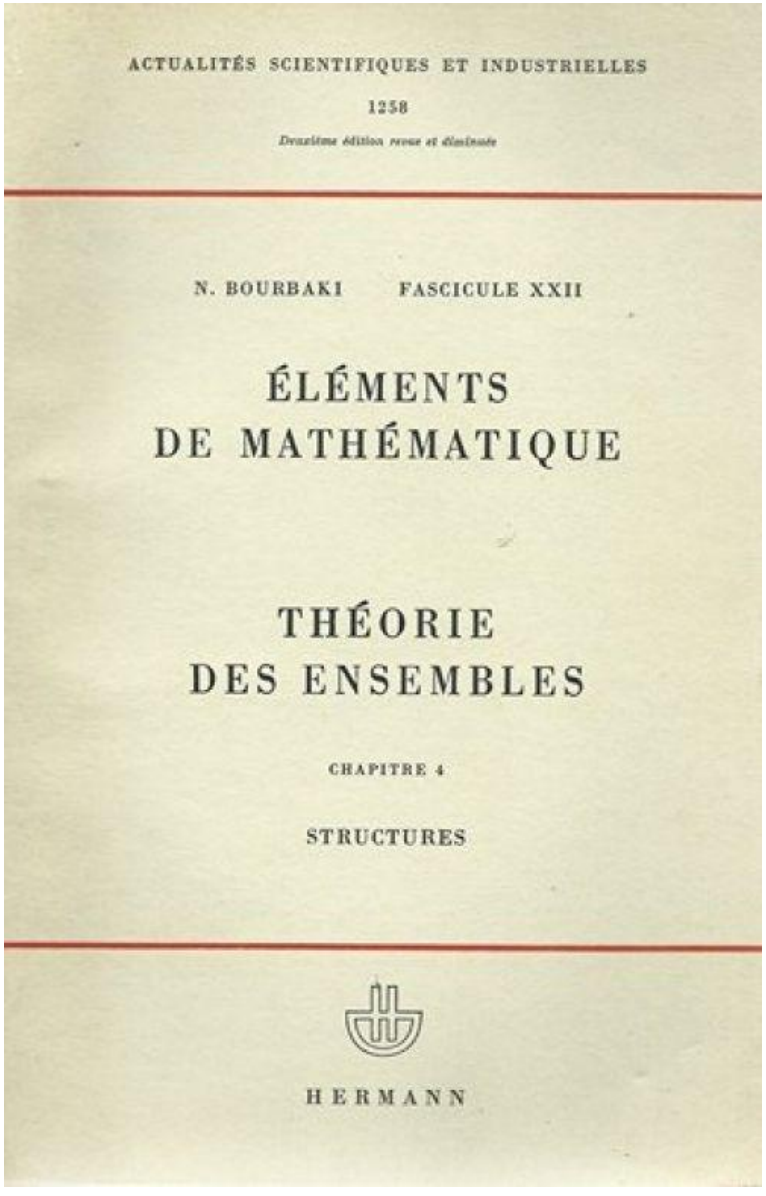
For a summary, see: Kathryn Edwards, 'Richard Dedekind (1831–1916)', *Mathematics Today* 52(1) (Feb 2016) 212–215

Set theory in our lives

Set theory as an effective language for mathematics:

- Set-builder notation
- Unification of ideas concerning functions and relations

Nicolas Bourbaki (1934–????)



Collective of French mathematicians who set out to reformulate mathematics on extremely formal, abstract, structural lines — the language of sets has a significant role to play.

Association des collaborateurs de Nicolas Bourbaki

SMP/New Math

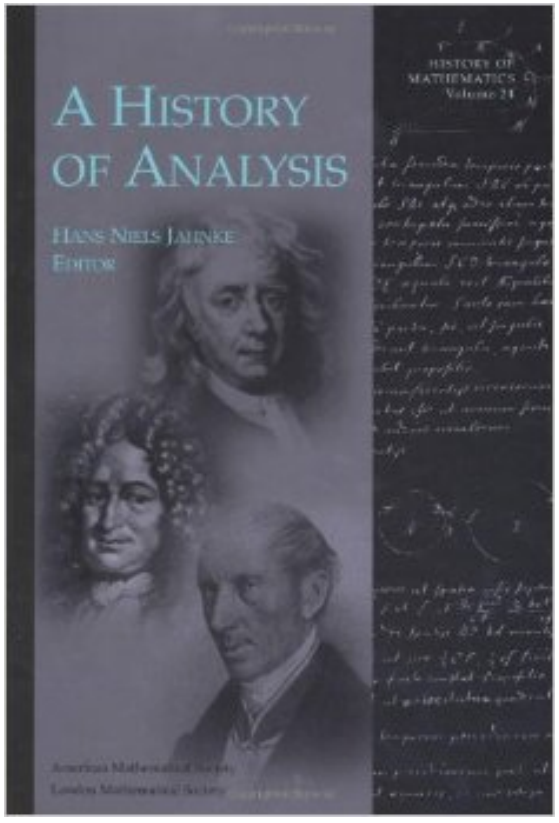
School Mathematics Project (UK)/New Mathematics (USA):

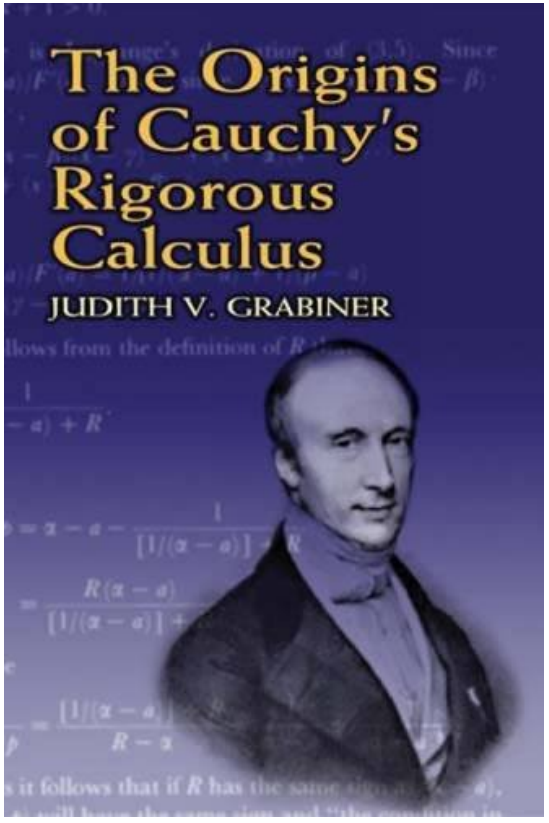
- Response to the launch of Sputnik I in 1957
- Traditional school arithmetic and geometry replaced by abstract algebra, matrices, symbolic logic, different bases, ... — in short, mathematical topics based on set theory
- Much debate — now usually regarded as a passing fad
- Tom Lehrer — 'New Math'

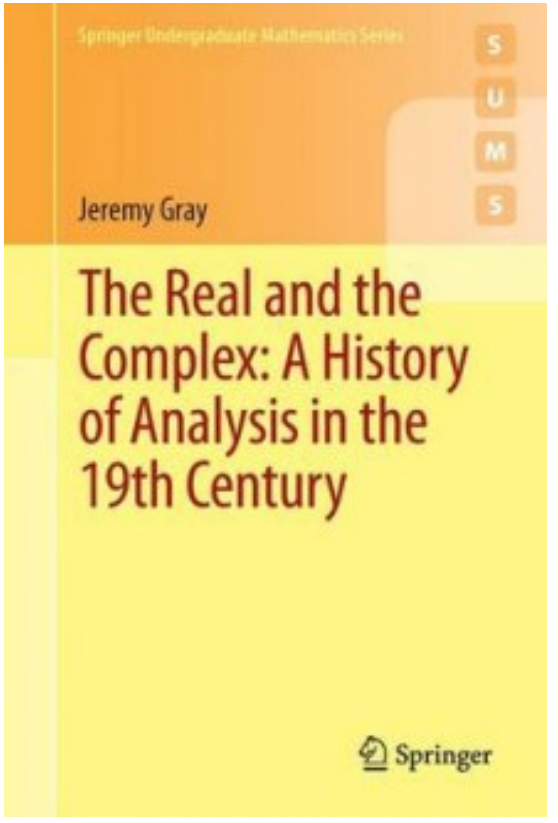
Conclusions

- Our modern perception of real numbers took well over 2000 years to crystallise, with geometric, arithmetic, set-theoretic intuitions to the fore at different times.
- The concept of set emerged at about the same time as the modern concept of real number, 1870–1890.
- This coincidence is no coincidence.

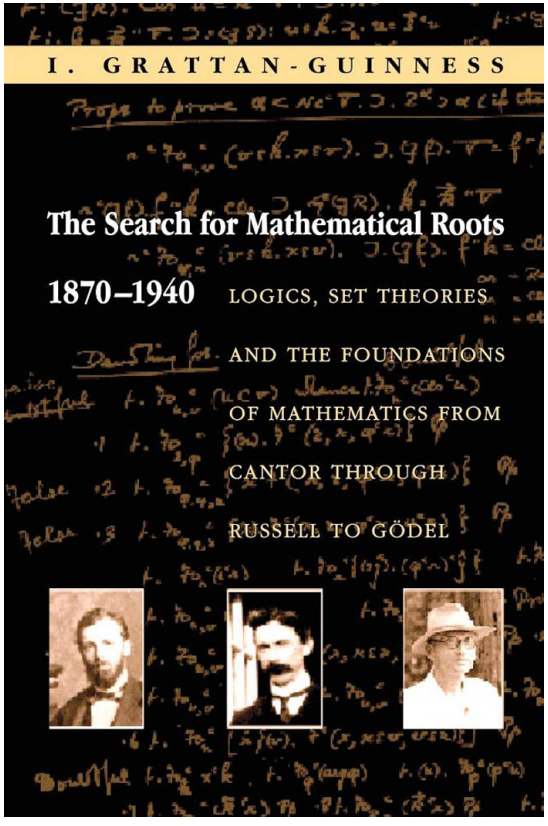
Further reading on the development of analysis...

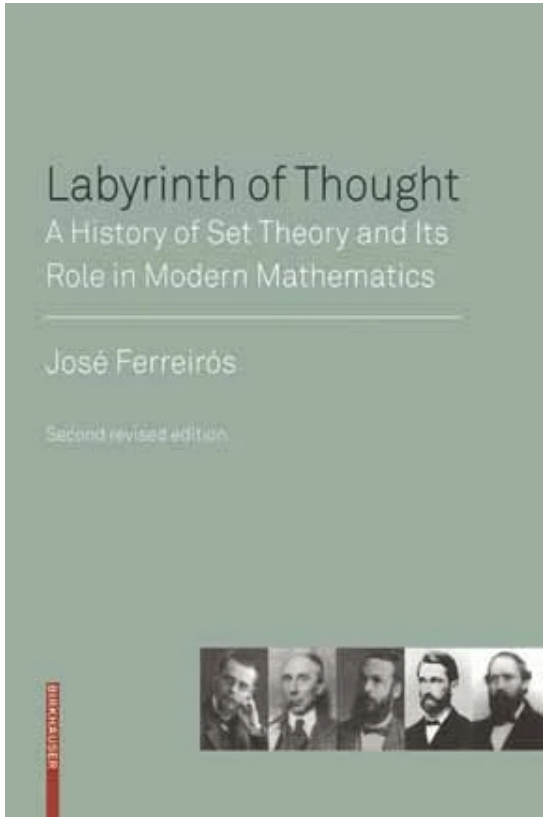


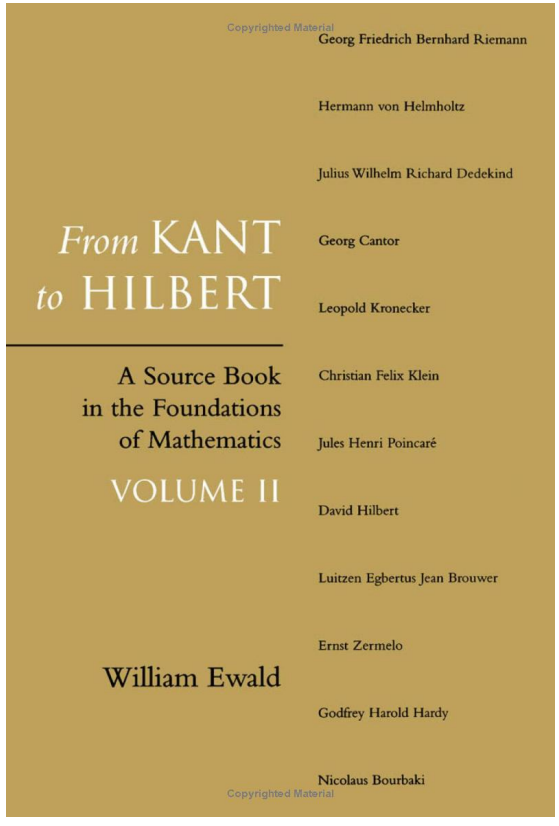



Springer Undergraduate Mathematics Series
SUMS
Jeremy Gray
The Real and the Complex: A History of Analysis in the 19th Century
Springer

...and on set theory and foundations







BO1 History of Mathematics
Lecture XIII
Complex analysis

MT 2022 Week 7

Summary

- Complex numbers: validity and representation
- Substitution of complex values for real
- Cauchy's contributions
- Riemann
- What *is* an analytic function?

Early ideas about complex numbers

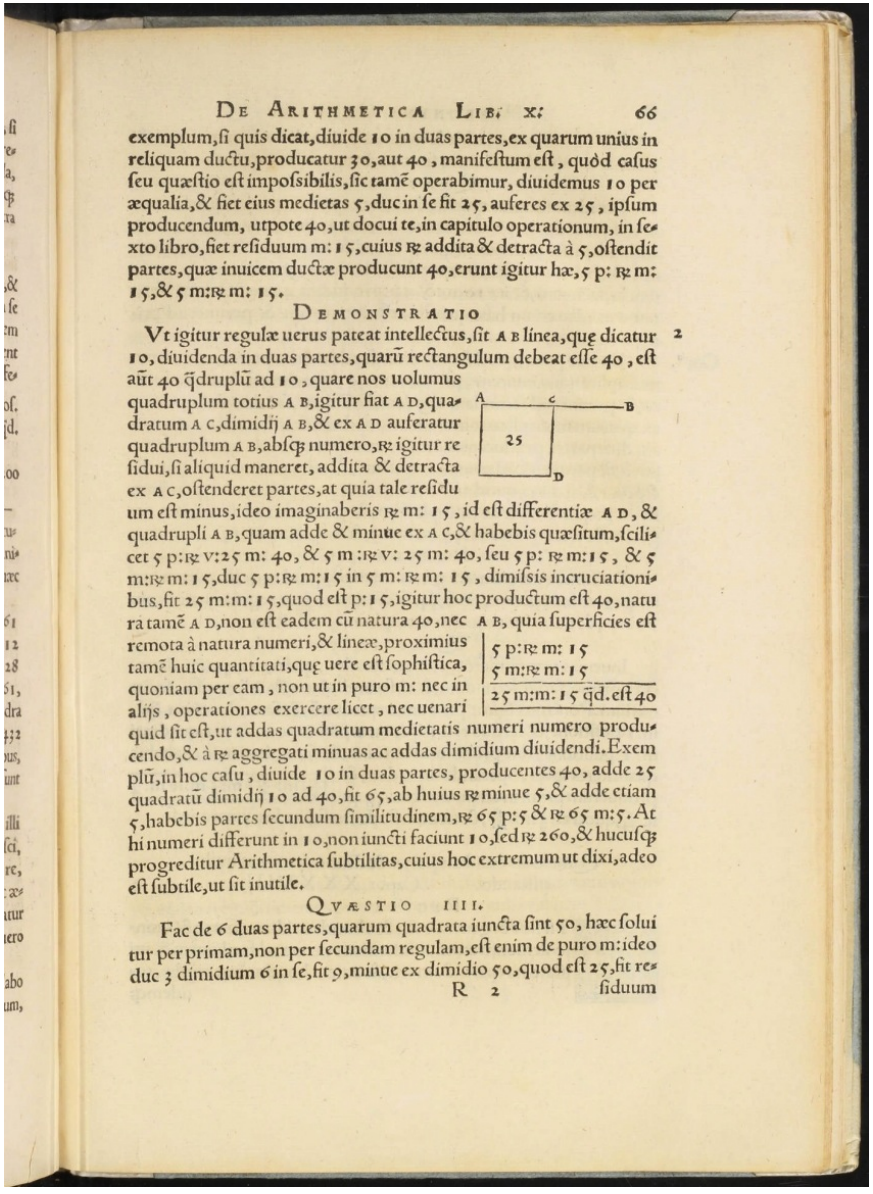
Before 1600, very faint beginnings:

- Cardano (1545) [from quadratics]
- Bombelli (1572) [from cubics]
- Harriot (c. 1600) [from quartics]

But:

> *For the most part such roots were ignored: negative roots were described merely as 'false', but complex roots as 'impossible'.* (*Mathematics emerging*, p. 459.)

Cardano and complex numbers



DE ARITHMETICA LIB. X. 66

exemplum, si quis dicat, diuide 10 in duas partes, ex quarum unius in reliquam ductu, producatur 30, aut 40, manifestum est, quòd casus seu quæstio est impossibilis, sic tamē operabimur, diuidemus 10 per æqualia, & fiet eius medietas 5, duc in se fit 25, auferes ex 25, ipsum producendum, utpote 40, ut docui te, in capitulo operationum, in sexto libro, fiet residuum m: 15, cuius ℞ addita & detracta à 5, ostendit partes, quæ inuicem ductæ producunt 40, erunt igitur hæ, 5 p: ℞ m: 15, & 5 m: ℞ m: 15.

DEMONSTRATIO

Vt igitur regulæ uerus pateat intellectus, sit A B linea, quę dicatur 10, diuidenda in duas partes, quarū rectangulum debeat esse 40, est aūt 40 q̄druplū ad 10, quare nos uolumus quadruplum totius A B, igitur fiat A D, quadratum A C, dimidij A B, & ex A D auferatur quadruplum A B, absq; numero, ℞ igitur residui, si aliquid maneret, addita & detracta ex A C, ostenderet partes, at quia tale residuum est minus, ideo imaginaberis ℞ m: 15, id est differentiæ A D, & quadrupli A B, quam adde & minue ex A C, & habebis quæsitum, scilicet 5 p: ℞ v: 25 m: 40, & 5 m: ℞ v: 25 m: 40, seu 5 p: ℞ m: 15, & 5 m: ℞ m: 15, duc 5 p: ℞ m: 15 in 5 m: ℞ m: 15, dimissis incruciationibus, fit 25 m: m: 15, quod est p: 15, igitur hoc productum est 40, natura tamē A D, non est eadem cū natura 40, nec A B, quia superficies est remota à natura numeri, & lineæ, proximius tamē huic quantitati, quę uere est sophistica, quoniam per eam, non ut in puro m: nec in alijs, operationes exercere licet, nec uenari quid sit est, ut addas quadratum medietatis numeri numero producendo, & à ℞ aggregati minuas ac addas dimidium diuidendi. Exemplū, in hoc casu, diuide 10 in duas partes, producentes 40, adde 25 quadratū dimidij 10 ad 40, fit 65, ab huius ℞ minue 5, & adde etiam 5, habebis partes secundum similitudinem, ℞ 65 p: 5 & ℞ 65 m: 5. At hi numeri differunt in 10, non iuncti faciunt 10, sed ℞ 260, & hucusq; progreditur Arithmetica subtilitas, cuius hoc extremum ut dixi, adeo est subtile, ut sit inutile.

5 p: ℞ m: 15
5 m: ℞ m: 15
25 m: m: 15 q̄d. est 40

QVÆSTIO IIII.

Fac de 6 duas partes, quarum quadrata iuncta sint 50, hæc solui tur per primam, non per secundam regulam, est enim de puro m: ideo duc 3 dimidium 6 in se, fit 9, minue ex dimidio 50, quod est 25, fit residuum

R 2

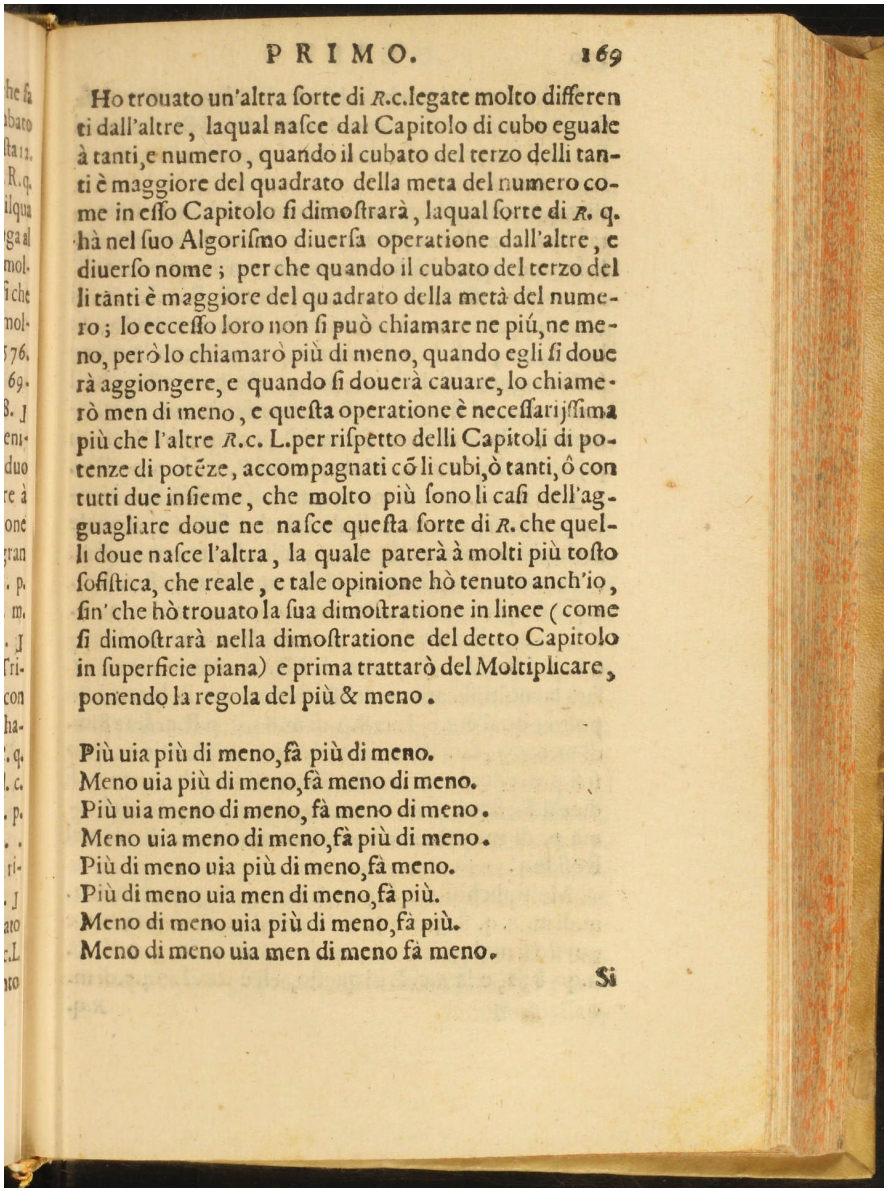
Problem: find two numbers that add to 10 and multiply to 40, i.e., solve an equation of the type 'square plus number equals thing'

Cardano noted that $5 + \sqrt{-15}$ and $5 - \sqrt{-15}$ solve the problem, "dismissis incruciationibus", meaning

"putting aside mental tortures",
or
"the cross-multiples having canceled out",
or
"the imaginary part being lost"

But regarded such ideas as absurd and useless

Bombelli and complex numbers



PRIMO. 169

Ho trouato un'altra ſorte di R.c.legate molto differenti dall'altre, laqual naſce dal Capitolo di cubo eguale à tanti,e numero, quando il cubato del terzo delli tanti è maggiore del quadrato della meta del numero come in eſſo Capitolo ſi dimoſtrarà, laqual ſorte di R. q. hà nel ſuo Algoriſmo diuerſa operatione dall'altre, e diuerſo nome; perche quando il cubato del terzo delli tanti è maggiore del quadrato della metà del numero; lo eccesſo loro non ſi può chiamare ne più,ne meno, però lo chiamarò più di meno, quando egli ſi douerà aggiongere, e quando ſi douerà cauare, lo chiamerò men di meno, e queſta operatione è neceſſariſſima più che l'altre R.c. L.per riſpetto delli Capitoli di potenze di potẽze, accompagnati cõ li cubi,ò tanti,ô con tutti due inſieme, che molto più ſono li caſi dell'agguagliare doue ne naſce queſta ſorte di R. che quelli doue naſce l'altra, la quale parerà à molti più toſto ſofiſtica, che reale, e tale opinione hò tenuto anch'io, fin' che hò trouato la ſua dimoſtratione in linee (come ſi dimoſtrarà nella dimoſtratione del detto Capitolo in ſuperficie piana) e prima trattarò del Moltiplicare, ponendo la regola del più & meno.

Più uia più di meno,fà più di meno.
Meno uia più di meno,fà meno di meno.
Più uia meno di meno, fà meno di meno.
Meno uia meno di meno,fà più di meno.
Più di meno uia più di meno,fà meno.
Più di meno uia men di meno,fà più.
Meno di meno uia più di meno,fà più.
Meno di meno uia men di meno fà meno.

Si

"Another sort of cube root much different from the former . . ."

Systematic rules:

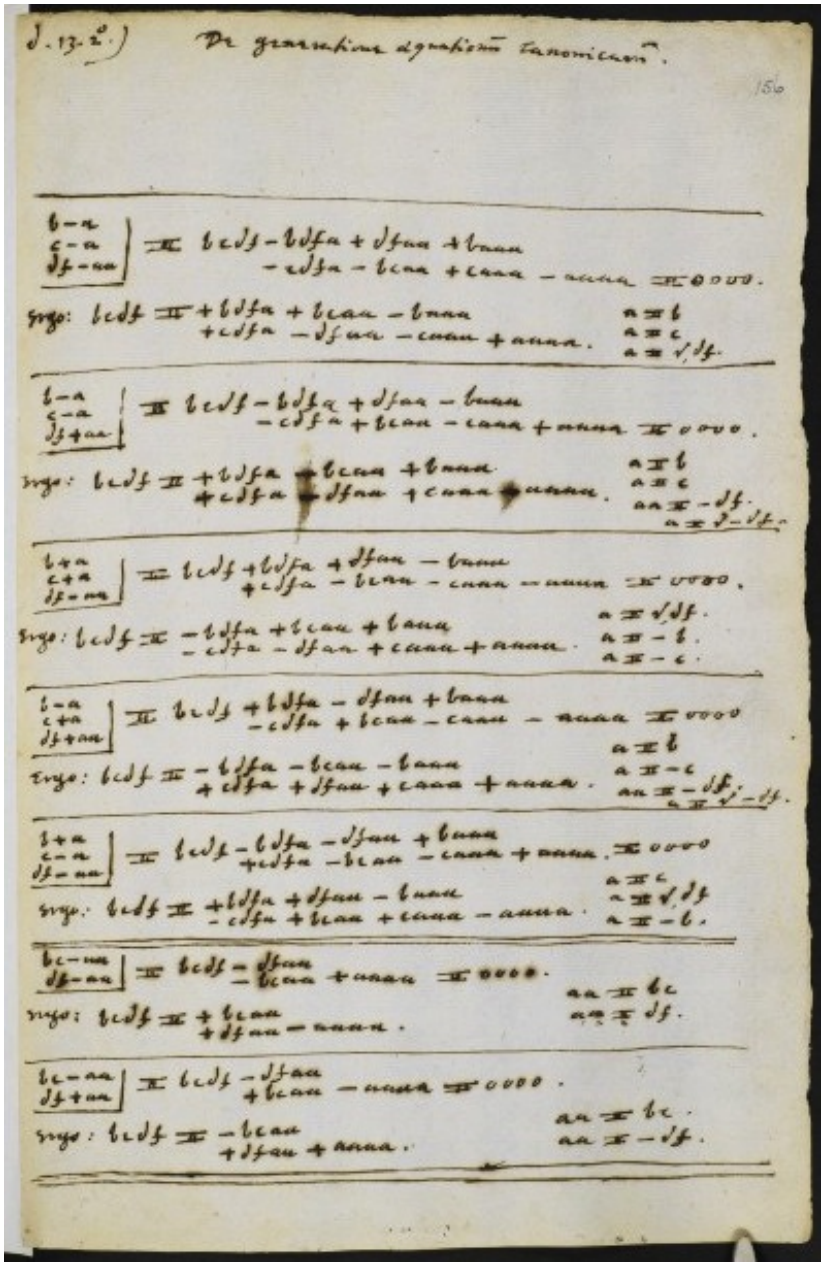
più di meno via più di meno, fà meno $(\sqrt{-1} \times \sqrt{-1} = -1)$

meno di meno via più di meno, fà più $(-\sqrt{-1} \times \sqrt{-1} = 1)$

But complex numbers were not admitted as solutions of equations — they could appear in calculations, provided they cancelled out by the end

Complex numbers justified through practical use?

Harriot and complex numbers



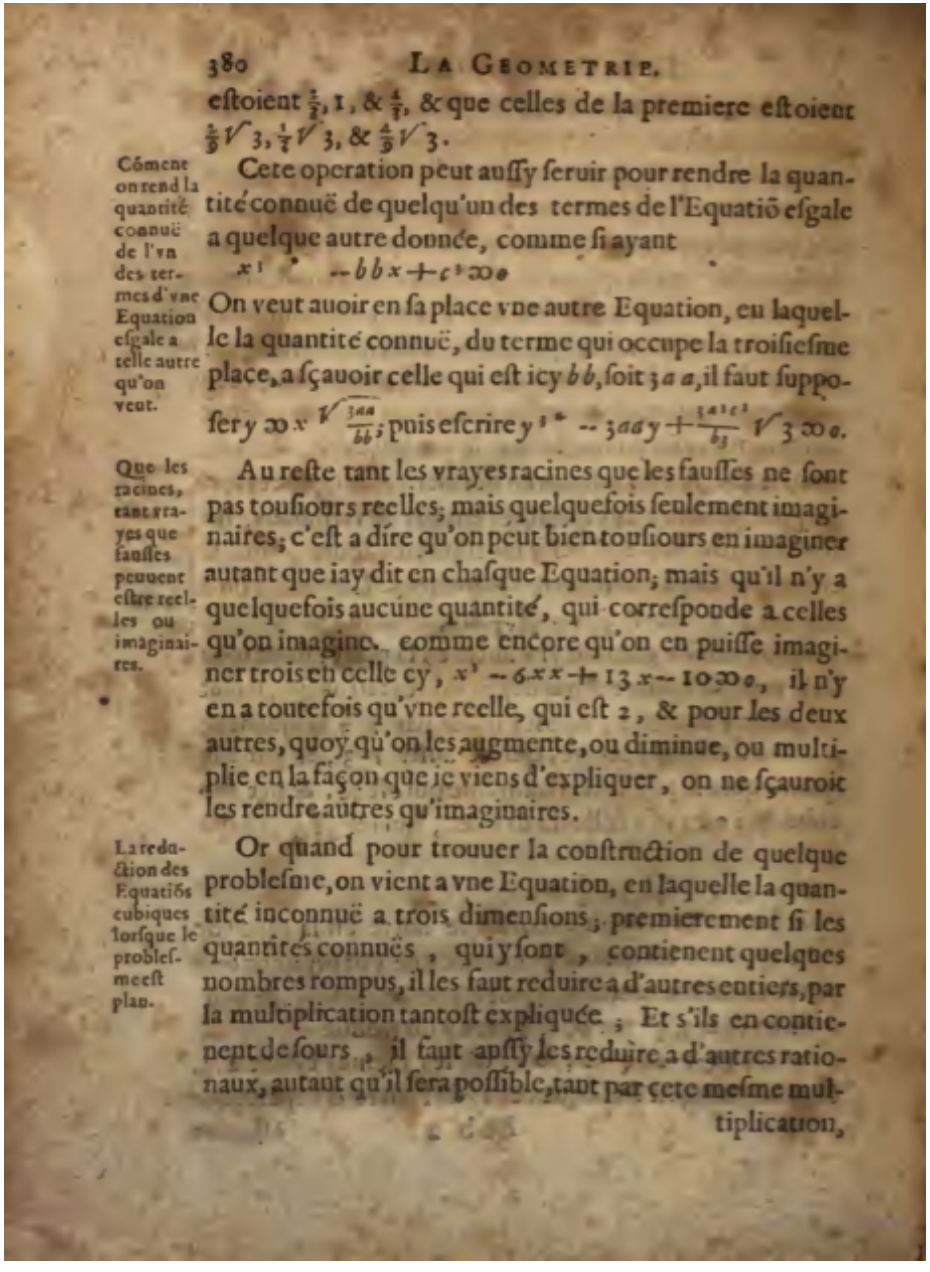
Add MS 6783 f. 156

Unpublished manuscripts contain systematic treatment of complex roots of equations — but these were removed by his editors

Cf. Harriot's *Artis analyticae praxis* (1931), pp. 14–15; see:

Muriel Seltman & Robert Goulding, *Thomas Harriot's Artis analyticae praxis: an English translation with commentary*, Springer, 2007

Descartes and ‘imaginaries’



380 LA GEOMETRIE.

estoient $\frac{1}{2}$, 1, & $\frac{4}{3}$, & que celles de la premiere estoient $\frac{2}{3}\sqrt{3}$, $\frac{1}{3}\sqrt{3}$, & $\frac{4}{3}\sqrt{3}$.

Comment on rend la quantité connuë de l'vn des termes d'vne Equation esgale a telle autre qu'on veut.

Cete operation peut aussy seruir pour rendre la quantité connuë de quelqu'un des termes de l'Equatiõ esgale a quelque autre donnée, comme si ayant $x^3 \propto -bbx+c^3$ on veut auoir en sa place vne autre Equation, en laquelle la quantité connuë, du terme qui occupe la troisiesme place, a sçauoir celle qui est icy bb, soit $3aa$, il faut supposer $y \propto x \sqrt{\frac{3aa}{bb}}$; puis escrire $y^3 \propto 3aay + \frac{3a^3c^3}{b^3}\sqrt{3} \propto o$.

Que les racines, tant vrayes que fausses peuuent estre reelles ou imaginaires.

Au reste tant les vrayes racines que les fausses ne sont pas touſiours reelles; mais quelquefois seulement imaginaires; c'est a dire qu'on peut bien touſiours en imaginer autant que iay dit en chasque Equation; mais qu'il n'y a quelquefois aucune quantité, qui corresponde a celles qu'on imagine. comme encore qu'on en puisse imaginer trois en celle cy, $x^3 - 6xx + 13x - 10 \propto o$, il n'y en a toutefois qu'vne reelle, qui est 2, & pour les deux autres, quoy qu'on les augmente, ou diminue, ou multiplie en la façon que je viens d'expliquer, on ne sçauroit les rendre autres qu'imaginaires.

La reduction des Equatiõs cubiques lorsque le probleſme est plan.

Or quand pour trouuer la construction de quelque probleſme, on vient a vne Equation, en laquelle la quantité inconnuë a trois dimensions; premierement si les quantités connuës, qui y sont, contienent quelques nombres rompus, il les faut reduire a d'autres entiers, par la multiplication tantost expliquée; Et s'ils en contienent de sours, il faut aussy les reduire a d'autres rationaux, autant qu'il sera possible, tant par cete meſme multiplication,

La géométrie (1637):

introduced the term ‘imaginaire’ — meant to be derogatory?

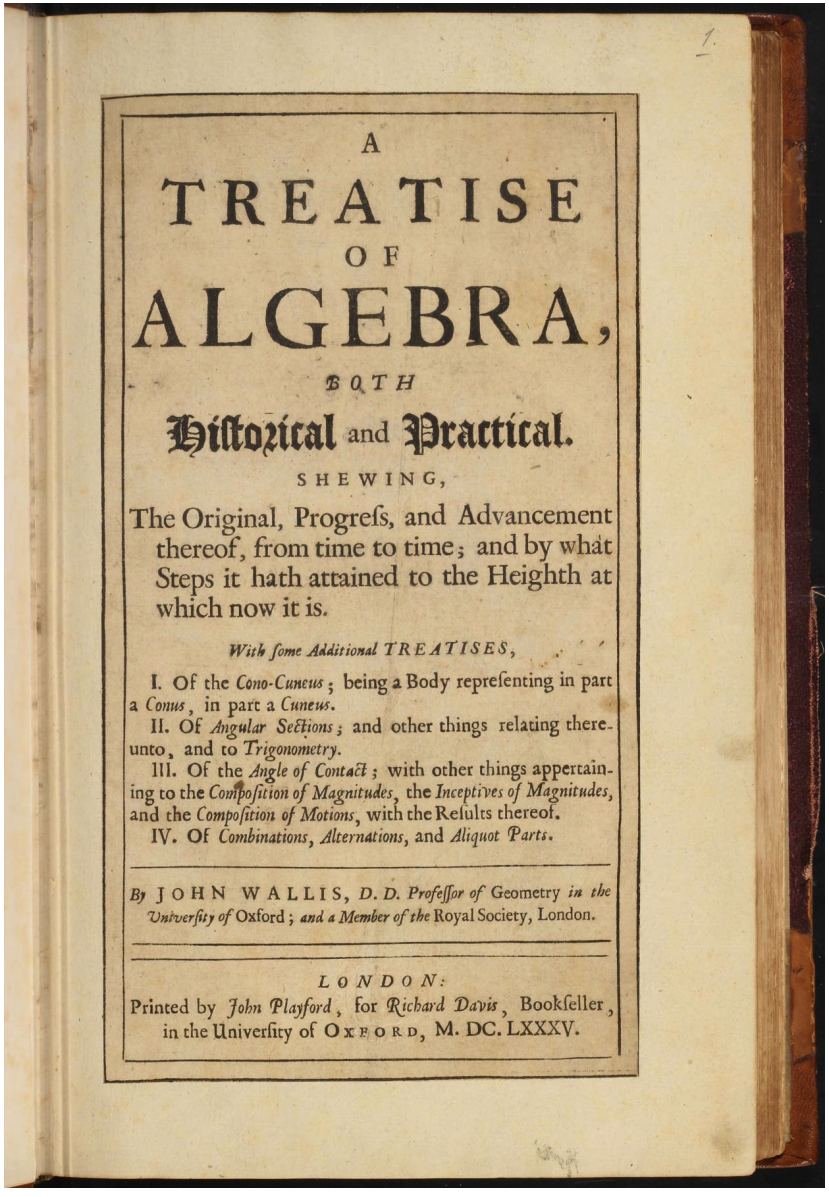
Didn’t regard them as numbers

Ideas about complex numbers in the later 17th century

John Wallis, *A treatise of algebra* (1685): complex numbers based on insights derived from

- Euclidean geometry
- trigonometry
- properties of conics

(See: *Mathematics emerging*, §15.1.1.)



A
TREATISE
OF
ALGEBRA,
BOTH
Hiſtorical and Practical.
SHEWING,
The Original, Progreſs, and Advancement thereof, from time to time; and by what Steps it hath attained to the Heighth at which now it is.

With ſome Additional TREATISES,

I. Of the *Cono-Cuneus*; being a Body repreſenting in part a *Conus*, in part a *Cuneus*.
II. Of *Angular Sections*; and other things relating thereunto, and to *Trigonometry*.
III. Of the *Angle of Contact*; with other things appertaining to the *Compoſition of Magnitudes*, the *Inceptives of Magnitudes*, and the *Compoſition of Motions*, with the Reſults thereof.
IV. Of *Combinations*, *Alternations*, and *Aliquot Parts*.

By JOHN WALLIS, *D. D. Profeſſor of* Geometry *in the Univerſity of* Oxford; *and a Member of the* Royal Society, London.

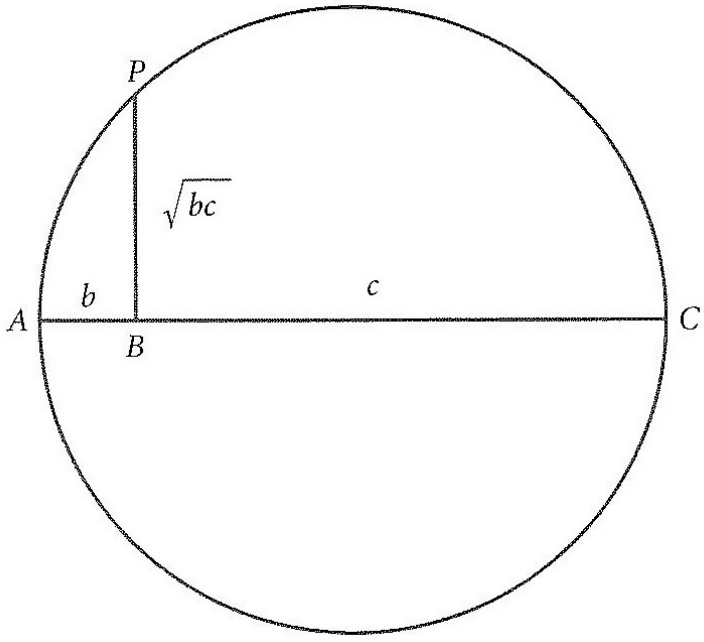
LONDON:
Printed by *John Playford*, for *Richard Davis*, Bookſeller, in the Univerſity of OXFORD, M. DC. LXXXV.

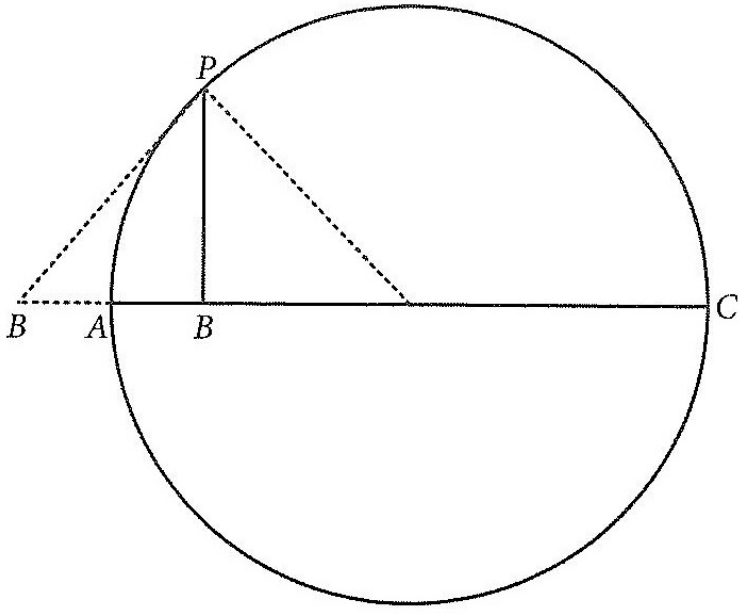
Wallis: justification of imaginary numbers

- A man starts at A and walks 5 yds to B, then retreats 2 yds to C: overall, he has covered 3 yds. If he instead retreats 8 yds to D, then we may say that he has covered -3 yds.
- Somewhere on the seashore, we gain 26 units of land from the sea, but lose 10 units. Thus, we have gained 16 units overall; if this is a perfect square, then it has side 4 units of length.
- If instead we lose 26 units of land, but gain 10, then we have lost 16 units overall, or gained -16. The area in question (assumed to be a square) might therefore be viewed as having side $\sqrt{-16}$.

(see: Leo Corry, *A brief history of numbers*, OUP, 2015, pp. 184–185)

Wallis: imaginary numbers as geometric means



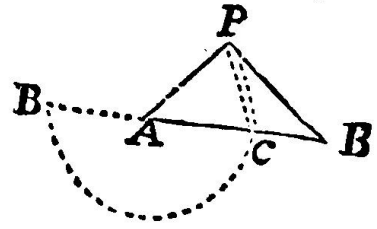


(see: Leo Corry, *A brief history of numbers*, OUP, 2015, pp. 185–186)

"A new Impossibility in Algebra"

John Wallis, *A treatise of algebra*, p. 267 'Of negative squares':
... requires a new Impossibility in Algebra

Suppoſe again, A P = 15, P C = 12, (and therefore A C = √ : 225 − 144 : = √ 81 = 9,) P B = 20 (and therefore B C = √ : 400 − 144 : = √ 256 = + 16, or − 16 :) Then is A B = 9 + 16 = 25, or A B = 9 − 16 = − 7. The one Affirmative, the other Negative. (The ſame values would be, but with contrary Signs, if we take A C = √ 81 = − 9 : That is, A B = − 9 + 16 = + 7, A B = − 9 − 16 = − 25.)



⋮

Which gives indeed (as before) a double value of A B, √ 175, + √ − 81, and √ 175, − √ − 81 : But ſuch as requires a new Impoſſibility in Algebra, (which in Lateral Equations doth not happen ;) not that of a Negative Root, or a Quantity leſs than nothing ; (as before,) but the Root of a Negative Square. Which in ſtrictneſs of ſpeech, cannot be : ſince that no Real Root (Affirmative or Negative,) being Multiplied into itſelf, will make a Negative Square.

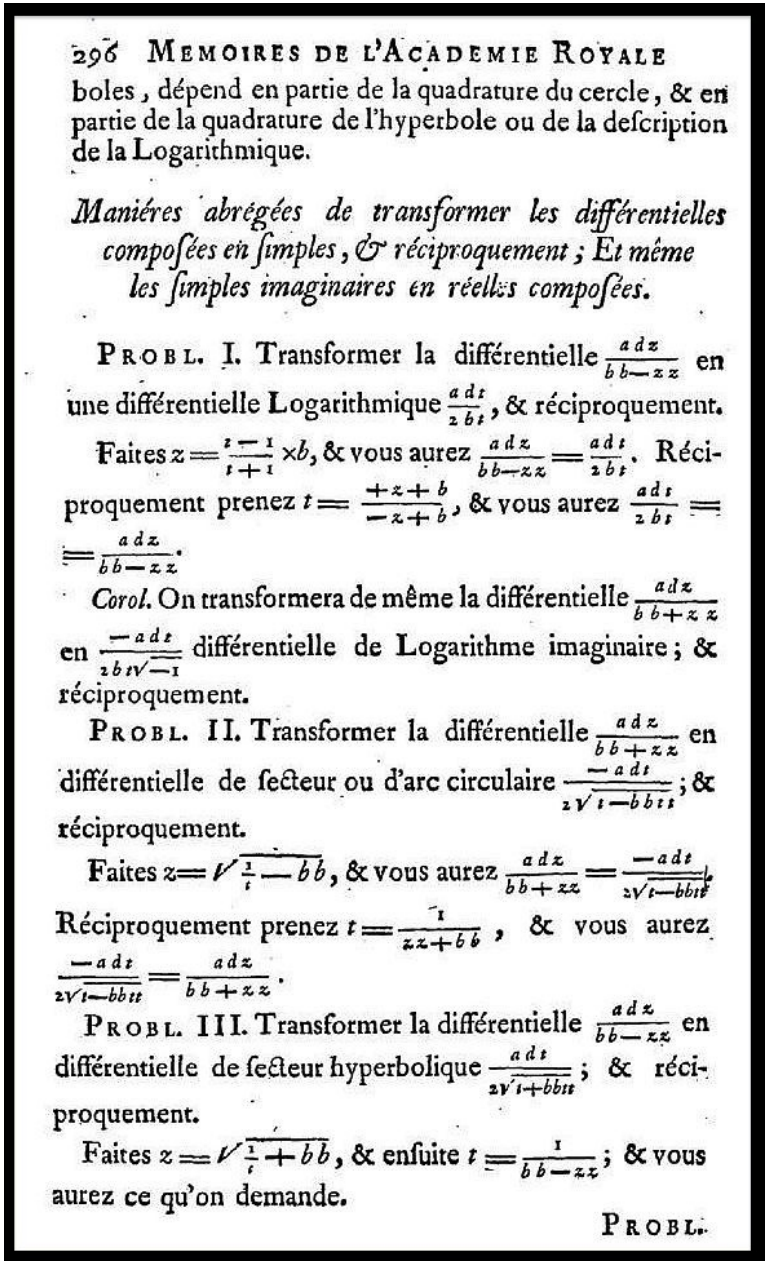
Complex numbers in the 18th century (1)

Nature remained unclear:

"that amphibian between being and not-being, which we call the imaginary root of negative unity" (Leibniz, 1702)

But complex numbers were increasingly being used ...

Complex numbers in the 18th century (2)



296 MEMOIRES DE L'ACADEMIE ROYALE

boles, dépend en partie de la quadrature du cercle, & en partie de la quadrature de l'hyperbole ou de la deſcription de la Logarithmique.

Maniéres abrégées de transformer les différentielles compoſées en ſimples, & réciproquement; Et même les ſimples imaginaires en réelles compoſées.

PROBL. I. Transformer la différentielle $\frac{adz}{bb-zz}$ en une différentielle Logarithmique $\frac{adt}{2bt}$, & réciproquement.

Faites $z=\frac{t-1}{t+1}\times b$, & vous aurez $\frac{adz}{bb-zz}=\frac{adt}{2bt}$. Réciproquement prenez $t=\frac{+z+b}{-z+b}$, & vous aurez $\frac{adt}{2bt}=\frac{adz}{bb-zz}$.

Corol. On transformera de même la différentielle $\frac{adz}{bb+zz}$ en $\frac{-adt}{2bt\sqrt{-1}}$ différentielle de Logarithme imaginaire; & réciproquement.

PROBL. II. Transformer la différentielle $\frac{adz}{bb+zz}$ en différentielle de ſecteur ou d'arc circulaire $\frac{-adt}{2\sqrt{t-bbtt}}$; & réciproquement.

Faites $z=\sqrt{\frac{1}{t}-bb}$, & vous aurez $\frac{adz}{bb+zz}=\frac{-adt}{2\sqrt{t-bbtt}}$. Réciproquement prenez $t=\frac{1}{zz+bb}$, & vous aurez $\frac{-adt}{2\sqrt{t-bbtt}}=\frac{adz}{bb+zz}$.

PROBL. III. Transformer la différentielle $\frac{adz}{bb-zz}$ en différentielle de ſecteur hyperbolique $\frac{adt}{2\sqrt{t+bbtt}}$; & réciproquement.

Faites $z=\sqrt{\frac{1}{t}+bb}$, & enſuite $t=\frac{1}{bb-zz}$; & vous aurez ce qu'on demande.

PROBL.

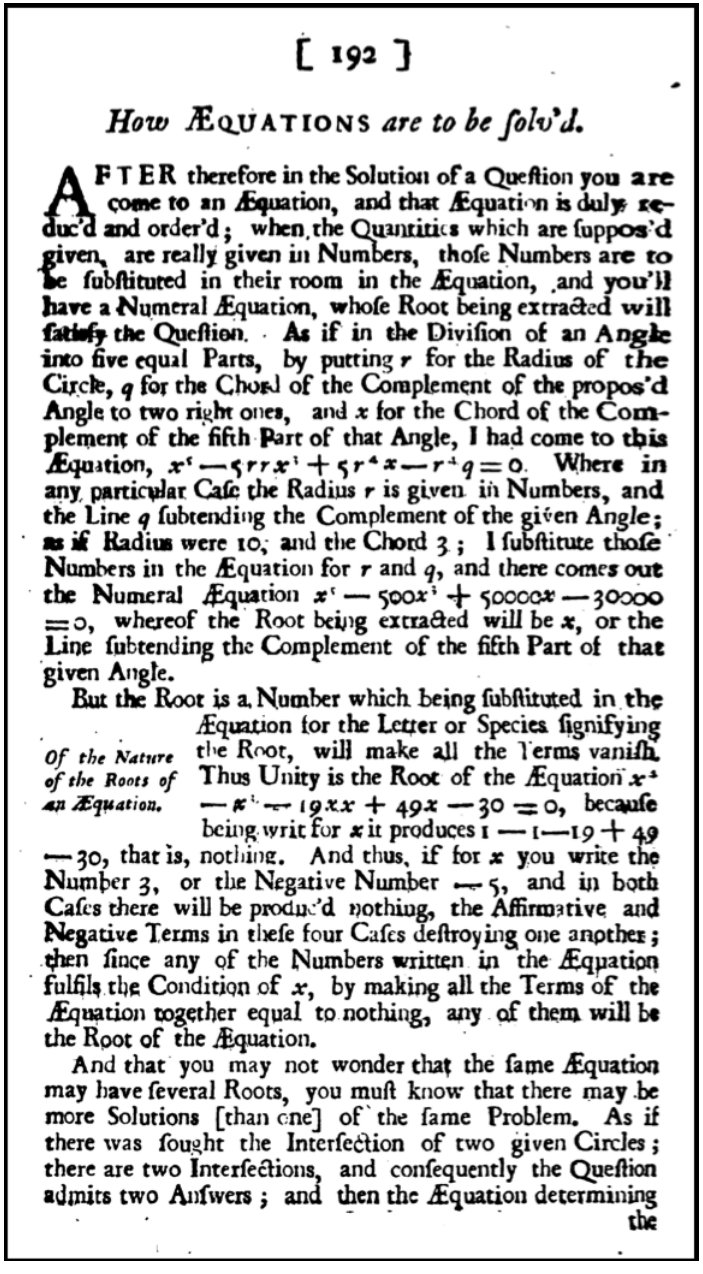
Johann Bernoulli, 'Solution d'un problème concernant le calcul intégrale, ...', *Mémoires de l'Académie royale des sciences*, 1702:

by making the substitution $z=\sqrt{\frac{1}{t}-bb}$, transform the differential $\frac{adz}{bb+zz}$ into $\frac{-adt}{2bt\sqrt{-1}}$

No worries about the validity of switching between real and complex integrals

(See *Mathematics emerging*, §15.2.1)

Complex numbers in the 18th century (3)



[192]

How ÆQUATIONS *are to be solv'd.*

AFTER therefore in the Solution of a Question you are come to an Æquation, and that Æquation is duly reduc'd and order'd; when the Quantities which are suppos'd given, are really given in Numbers, those Numbers are to be substituted in their room in the Æquation, and you'll have a Numeral Æquation, whose Root being extracted will satisfy the Question. As if in the Division of an Angle into five equal Parts, by putting r for the Radius of the Circle, q for the Chord of the Complement of the propos'd Angle to two right ones, and x for the Chord of the Complement of the fifth Part of that Angle, I had come to this Æquation, $x^5 - 5rrx^3 + 5r^4x - r^4q = 0$. Where in any particular Case the Radius r is given in Numbers, and the Line q subtending the Complement of the given Angle; as if Radius were 10, and the Chord 3; I substitute those Numbers in the Æquation for r and q, and there comes out the Numeral Æquation $x^5 - 500x^3 + 50000x - 30000 = 0$, whereof the Root being extracted will be x, or the Line subtending the Complement of the fifth Part of that given Angle.

Of the Nature of the Roots of an Æquation.

But the Root is a Number which being substituted in the Æquation for the Letter or Species signifying the Root, will make all the Terms vanish. Thus Unity is the Root of the Æquation $x^4 - x^3 - 19xx + 49x - 30 = 0$, because being writ for x it produces $1 - 1 - 19 + 49 - 30$, that is, nothing. And thus, if for x you write the Number 3, or the Negative Number -5, and in both Cases there will be produc'd nothing, the Affirmative and Negative Terms in these four Cases destroying one another; then since any of the Numbers written in the Æquation fulfils the Condition of x, by making all the Terms of the Æquation together equal to nothing, any of them will be the Root of the Æquation.

And that you may not wonder that the same Æquation may have several Roots, you must know that there may be more Solutions [than one] of the same Problem. As if there was sought the Intersection of two given Circles; there are two Intersections, and consequently the Question admits two Answers; and then the Æquation determining the

Isaac Newton, *Universal Arithmetick*, 1728:

p. 195: "it is just that the Roots of Equations should be often impossible, lest they should exhibit the cases of Problems that are impossible as if they are possible" — complex numbers as an indicator of real-world solvability of problems

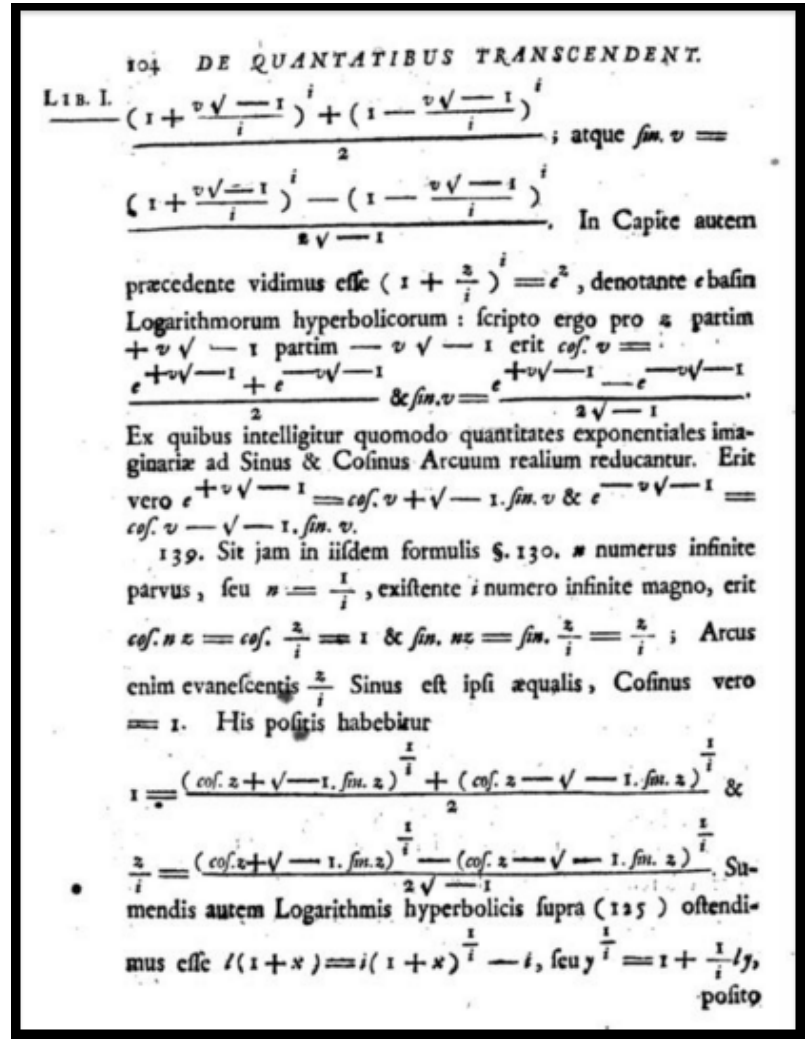
Complex numbers in the 18th century (4)

Leonhard Euler also used them freely: e.g., in *Introductio in analysin infinitorum*, 1748, §138:

$$e^{+v\sqrt{-1}} = \cos .v + \sqrt{-1}. \sin .v$$

$$e^{-v\sqrt{-1}} = \cos .v - \sqrt{-1}. \sin .v$$

(See *Mathematics emerging*, §9.2.3)



104 DE QUANTATIBUS TRANSCENDENT.

LIB. I.

$\frac{(1+\frac{v\sqrt{-1}}{i})^i + (1-\frac{v\sqrt{-1}}{i})^i}{2}$; atque $\sin. v = \frac{(1+\frac{v\sqrt{-1}}{i})^i - (1-\frac{v\sqrt{-1}}{i})^i}{2\sqrt{-1}}$. In Capite autem præcedente vidimus esse $(1+\frac{z}{i})^i = e^z$, denotante e basin Logarithmorum hyperbolicorum: scripto ergo pro z partim $+v\sqrt{-1}$ partim $-v\sqrt{-1}$ erit $\cos. v = \frac{e^{+v\sqrt{-1}} + e^{-v\sqrt{-1}}}{2}$ & $\sin. v = \frac{e^{+v\sqrt{-1}} - e^{-v\sqrt{-1}}}{2\sqrt{-1}}$.

Ex quibus intelligitur quomodo quantitates exponentiales imaginariæ ad Sinus & Cosinus Arcuum realium reducantur. Erit vero $e^{+v\sqrt{-1}} = \cos. v + \sqrt{-1}. \sin. v$ & $e^{-v\sqrt{-1}} = \cos. v - \sqrt{-1}. \sin. v$.

139. Sit jam in iisdem formulis §. 130. n numerus infinite parvus, seu $n = \frac{1}{i}$, existente i numero infinite magno, erit $\cos. nz = \cos. \frac{z}{i} = 1$ & $\sin. nz = \sin. \frac{z}{i} = \frac{z}{i}$; Arcus enim evanescentis $\frac{z}{i}$ Sinus est ipsi æqualis, Cosinus vero $= 1$. His positis habebitur

$1 = \frac{(\cos. z + \sqrt{-1}. \sin. z)^{\frac{1}{i}} + (\cos. z - \sqrt{-1}. \sin. z)^{\frac{1}{i}}}{2}$ &

$\frac{z}{i} = \frac{(\cos. z + \sqrt{-1}. \sin. z)^{\frac{1}{i}} - (\cos. z - \sqrt{-1}. \sin. z)^{\frac{1}{i}}}{2\sqrt{-1}}$. Sumendis autem Logarithmis hyperbolicis supra (125) ostendimus esse $l(1+x) = i(1+x)^{\frac{1}{i}} - i$, seu $y^{\frac{1}{i}} = 1 + \frac{1}{i} ly$, posito

The Fundamental Theorem of Algebra

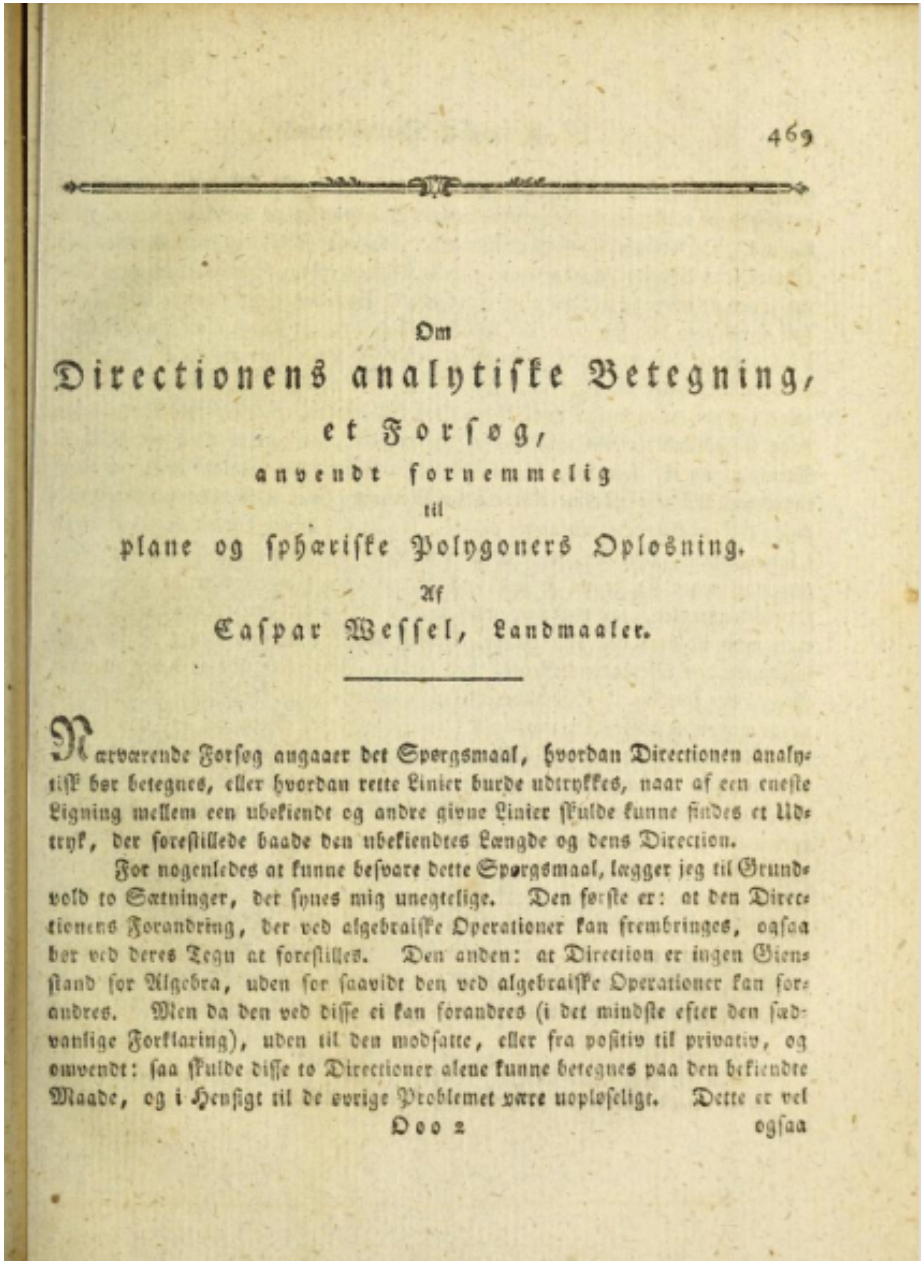
Every polynomial equation of degree n has exactly n roots.

- Early 17th century: known that an equation of degree n may have n roots
- During 17th century: complex numbers gradually admitted as roots
- 15 Sept 1759: Euler asserted theorem in a letter to Nicholas Bernoulli, but didn't prove it
- Mid/late 18th century: attempted proofs by Euler, d'Alembert, Lagrange, and others
- 1799: proof by Gauss in his doctoral dissertation, followed by several others
- 1806: new proof by Argand
- 1821: Argand's proof appears in Cauchy's *Cours d'analyse*

Gauss and complex numbers

"If this subject has hitherto been considered from the wrong viewpoint and thus enveloped in mystery and surrounded by darkness, it is largely an unsuitable terminology which should be blamed. Had $+1$, -1 and $\sqrt{-1}$, instead of being called positive, negative and imaginary (or worse still, impossible) unity, been given the names say, of direct, inverse and lateral unity, there would hardly have been any scope for such obscurity." (1831)

New ways of viewing complex numbers



469

Om

Directionens analytiske Betegning,

et Forsøg,

anvendt fornemmelig

til

plane og sphæriske Polygoners Oplosning.

Af

Caspar Wessel, Landmaaler.

Nærværende Forsøg angaaer det Spørgsmaal, hvordan Directionen analytisk bør betegnes, eller hvordan rette Linier burde udtrykkes, naar af een eneste Ligning mellem een ubekiendt og andre givne Linier skulde kunne findes et Udtryk, der forestillede baade den ubekiendtes Længde og dens Direction.

For nogenledes at kunne besvare dette Spørgsmaal, lægger jeg til Grundvold to Sætninger, der synes mig unegtelige. Den første er: at den Directionens Forandring, der ved algebraiske Operationer kan frembringes, ogsaa bør ved deres Tegn at forestilles. Den anden: at Direction er ingen Gienstand for Algebra, uden for saavidt den ved algebraiske Operationer kan forandres. Men da den ved disse ei kan forandres (i det mindste efter den sædvanlige Forklaring), uden til den modsatte, eller fra positiv til privativ, og omvendt: saa skulde disse to Directioner alene kunne betegnes paa den bekiendte Maade, og i Henseende til de øvrige Problemet være uopløseligt. Dette er vel

Caspar Wessel, 'Om Directionens analytiske Betegning ...' ['On the analytic representation of direction ...'], *Nye Samling af det Kongelige Danske Videnskabers Selskabs Skrifter*, 1799

Published in Danish — not well known

French translation published in 1897

New ways of viewing complex numbers

ESSAI

SUR UNE MANIÈRE DE REPRÉSENTER

LES QUANTITÉS IMAGINAIRES

DANS

LES CONSTRUCTIONS GÉOMÉTRIQUES,

PAR R. ARGAND.

2e ÉDITION

PRÉCÉDÉE D'UNE PRÉFACE

PAR M. J. HOÜEL

ET SUIVIE D'UN APPENDICE

Contenant des Extraits des *Annales de Gergonne*, relatifs à la question des imaginaires.

PARIS,

GAUTHIER-VILLARS, IMPRIMEUR-LIBRAIRE

DU BUREAU DES LONGITUDES, DE L'ÉCOLE POLYTECHNIQUE,

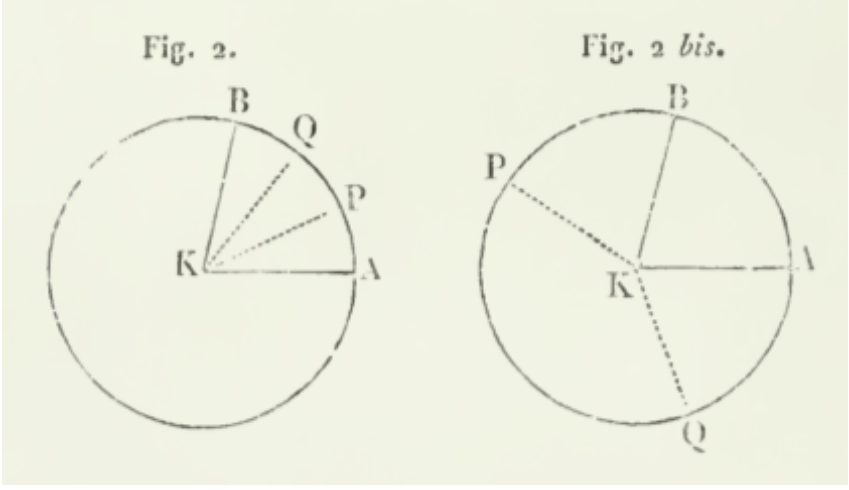
SUCCESSEUR DE MALLET-BACHELIER,

Quai des Augustins, 55.

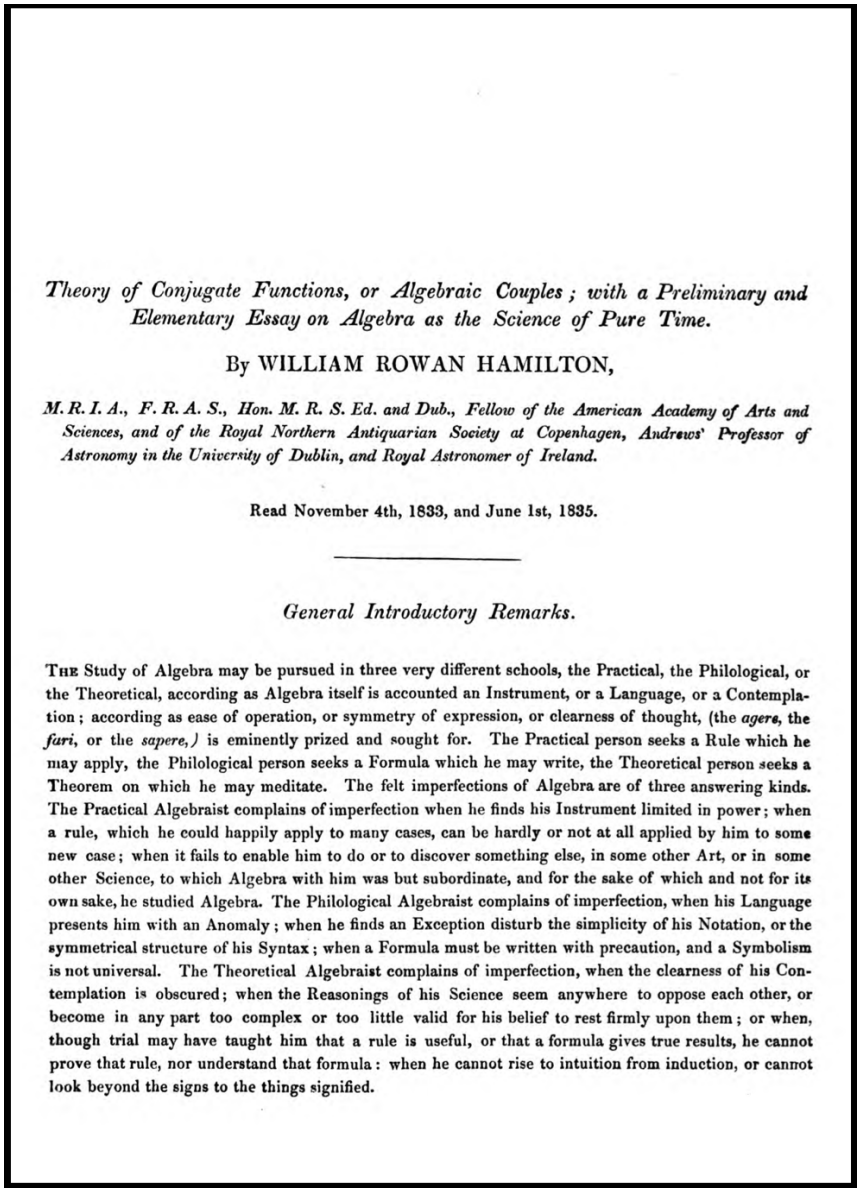
1874

(Tous droits réservés.)

Robert Argand, *Essay on a method of representing imaginary quantities* . . ., 1806



New ways of viewing complex numbers



Theory of Conjugate Functions, or Algebraic Couples; with a Preliminary and Elementary Essay on Algebra as the Science of Pure Time.

By WILLIAM ROWAN HAMILTON,

M. R. I. A., F. R. A. S., Hon. M. R. S. Ed. and Dub., Fellow of the American Academy of Arts and Sciences, and of the Royal Northern Antiquarian Society at Copenhagen, Andrews' Professor of Astronomy in the University of Dublin, and Royal Astronomer of Ireland.

Read November 4th, 1833, and June 1st, 1835.

General Introductory Remarks.

THE Study of Algebra may be pursued in three very different schools, the Practical, the Philological, or the Theoretical, according as Algebra itself is accounted an Instrument, or a Language, or a Contemplation; according as ease of operation, or symmetry of expression, or clearness of thought, (the *agere*, the *fari*, or the *sapere*,) is eminently prized and sought for. The Practical person seeks a Rule which he may apply, the Philological person seeks a Formula which he may write, the Theoretical person seeks a Theorem on which he may meditate. The felt imperfections of Algebra are of three answering kinds. The Practical Algebraist complains of imperfection when he finds his Instrument limited in power; when a rule, which he could happily apply to many cases, can be hardly or not at all applied by him to some new case; when it fails to enable him to do or to discover something else, in some other Art, or in some other Science, to which Algebra with him was but subordinate, and for the sake of which and not for its own sake, he studied Algebra. The Philological Algebraist complains of imperfection, when his Language presents him with an Anomaly; when he finds an Exception disturb the simplicity of his Notation, or the symmetrical structure of his Syntax; when a Formula must be written with precaution, and a Symbolism is not universal. The Theoretical Algebraist complains of imperfection, when the clearness of his Contemplation is obscured; when the Reasonings of his Science seem anywhere to oppose each other, or become in any part too complex or too little valid for his belief to rest firmly upon them; or when, though trial may have taught him that a rule is useful, or that a formula gives true results, he cannot prove that rule, nor understand that formula: when he cannot rise to intuition from induction, or cannot look beyond the signs to the things signified.

Transactions of the Royal Irish Academy, 1837

Complex numbers as ordered pairs subject to specified rules:

$$(a, b) \pm (c, d) = (a \pm c, b \pm d)$$

$$(a, b)(c, d) = (ac - bd, ad + bc)$$

$$\frac{(a, b)}{(c, d)} = \left(\frac{ac + bd}{c^2 + d^2}, \frac{bc - ad}{c^2 + d^2} \right)$$

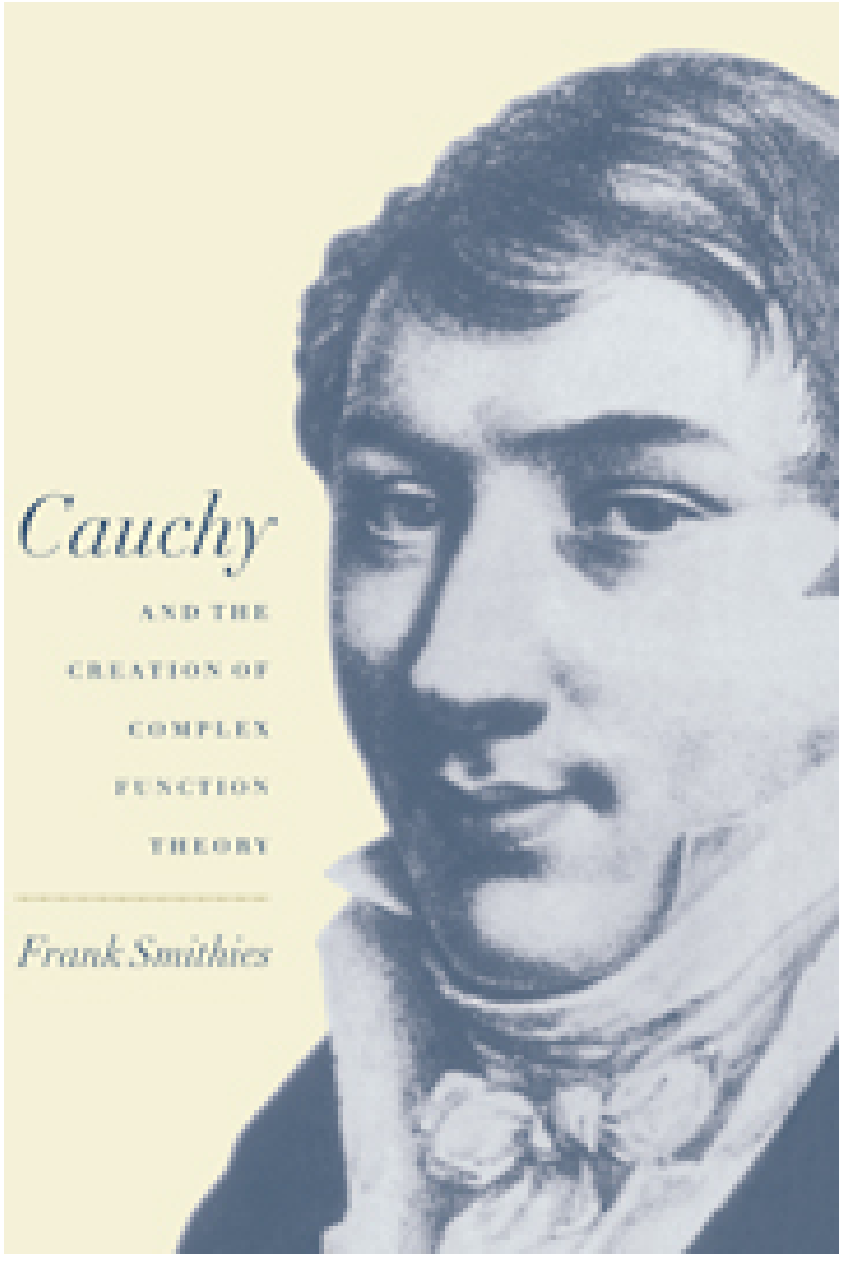
Led to the search for triples, and thence to quaternions

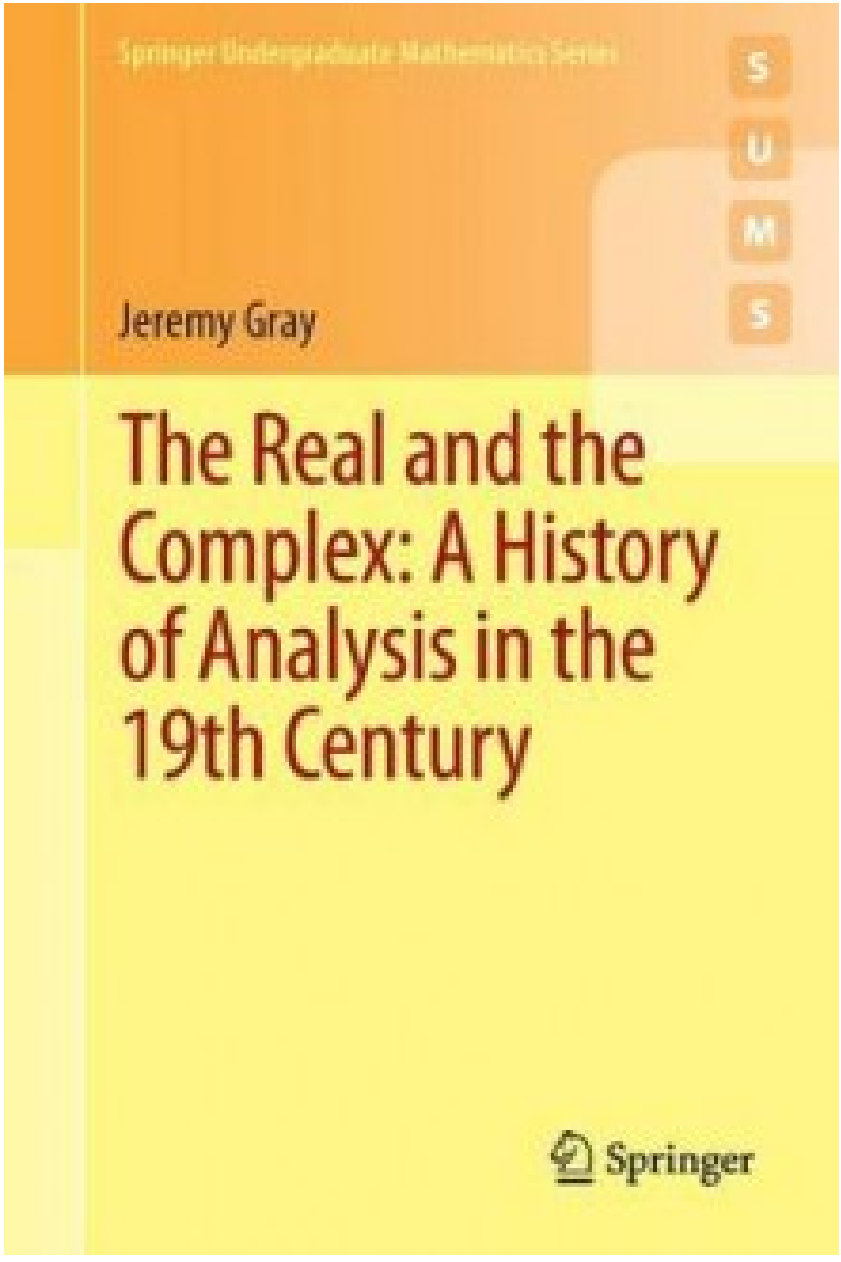
Complex analysis

The origins of complex analysis may be seen in early achievements by Johann Bernoulli, Euler, and others, using complex transformations to evaluate real integrals. But is substitution of complex variables for real variables permissible?

- Euler (posthumous, 1794): yes
- Laplace (1785, 1812): yes
- Poisson (1812): doubtful
- Cauchy (1814): inspired by Laplace, set to work on the problem

Sources for the origins of complex analysis




Springer Undergraduate Mathematics Series
SUMS
Jeremy Gray
The Real and the Complex: A History of Analysis in the 19th Century
Springer

Cauchy as 'creator' of complex analysis

Some of Cauchy's contributions to complex analysis:

- integration along paths and contours (1814) [1827]
- calculus of residues (1826)
- integral formulae (1831)
- inferences about Taylor series expansions
- applications to evaluation of difficult definite integrals of real functions

Cauchy's changing views of complex numbers and variables

At different times, Cauchy regarded complex numbers in different ways:

- as formal (numerical) expressions $a + b\sqrt{-1}$;
- geometrically;
- by reducing $i = \sqrt{-1}$ to a "real but indeterminate quantity"

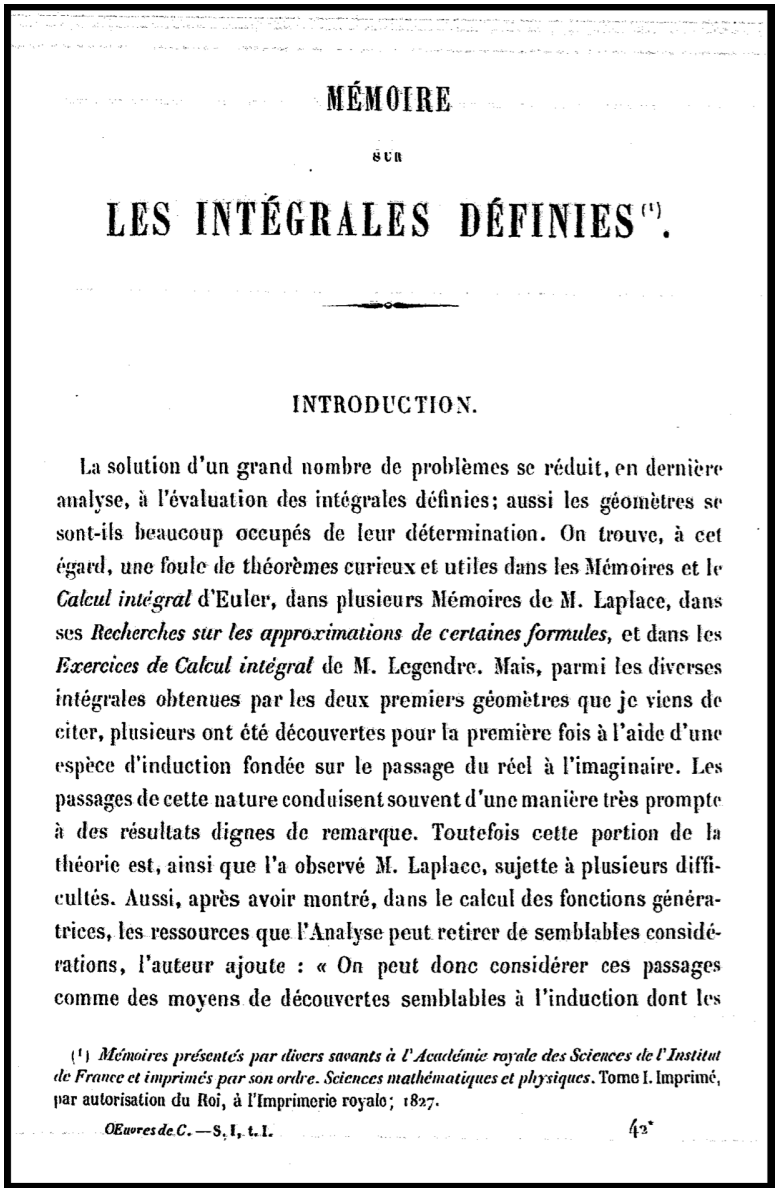
  This done, there is no need to torture the mind to discover what the symbolic sign $\sqrt{-1}$ *could represent* ...

  (in modern terms, Cauchy reduced complex arithmetic to calculations modulo $i^2 + 1$ in $\mathbb{R}[i]$)

Moreover, Cauchy's view of complex variables gradually shifted

- from quantities with two parts $x + y\sqrt{-1}$
- to single quantities z.

Cauchy's first 'Mémoire' (1814/1827)



MÉMOIRE

SUR

LES INTÉGRALES DÉFINIES[1].

INTRODUCTION.

La solution d'un grand nombre de problèmes se réduit, en dernière analyse, à l'évaluation des intégrales définies; aussi les géomètres se sont-ils beaucoup occupés de leur détermination. On trouve, à cet égard, une foule de théorèmes curieux et utiles dans les Mémoires et le *Calcul intégral* d'Euler, dans plusieurs Mémoires de M. Laplace, dans ses *Recherches sur les approximations de certaines formules*, et dans les *Exercices de Calcul intégral* de M. Legendre. Mais, parmi les diverses intégrales obtenues par les deux premiers géomètres que je viens de citer, plusieurs ont été découvertes pour la première fois à l'aide d'une espèce d'induction fondée sur le passage du réel à l'imaginaire. Les passages de cette nature conduisent souvent d'une manière très prompte à des résultats dignes de remarque. Toutefois cette portion de la théorie est, ainsi que l'a observé M. Laplace, sujette à plusieurs difficultés. Aussi, après avoir montré, dans le calcul des fonctions génératrices, les ressources que l'Analyse peut retirer de semblables considérations, l'auteur ajoute : « On peut donc considérer ces passages comme des moyens de découvertes semblables à l'induction dont les

[1] *Mémoires présentés par divers savants à l'Académie royale des Sciences de l'Institut de France et imprimés par son ordre. Sciences mathématiques et physiques.* Tome I. Imprimé, par autorisation du Roi, à l'Imprimerie royale; 1827.

OEuvres de C. — S. I, t. I. 42*

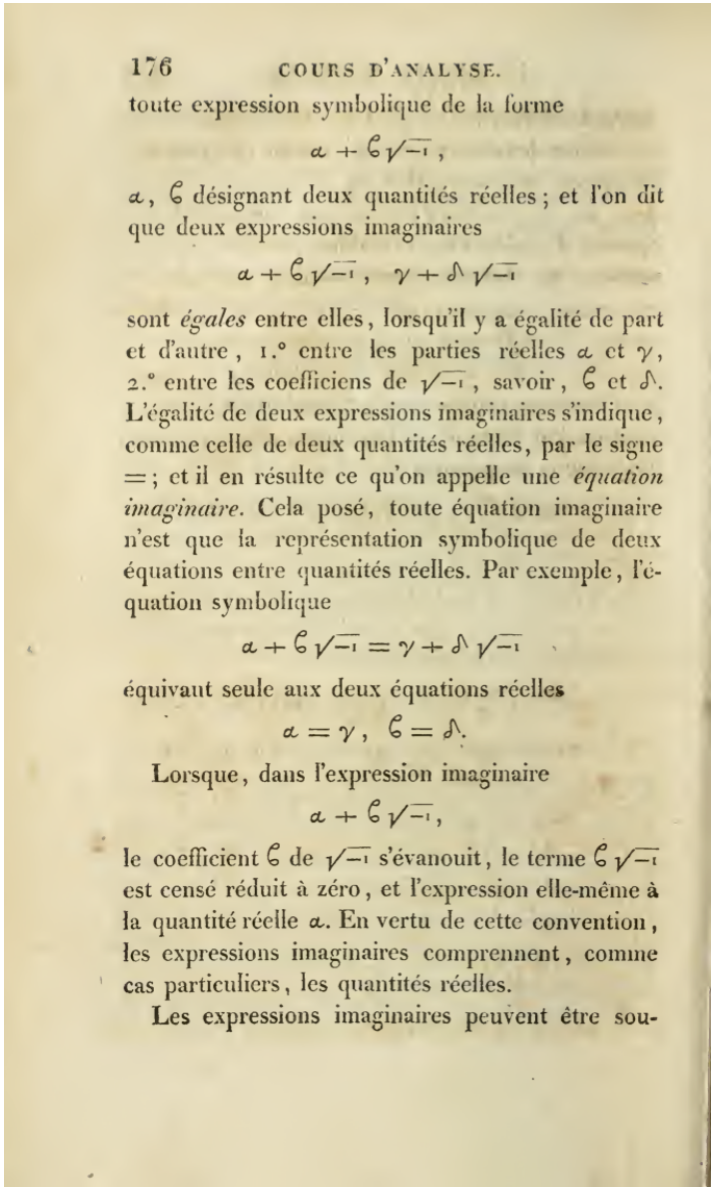
Cited Laplace's concerns about the solution of integrals by "the passage from the real to the imaginary"

First part: evaluation of improper integrals, such as

$$\int_{-\infty}^{\infty} \frac{\cos x}{1+x^2}\, dx = \frac{\pi}{e}$$

Noted Cauchy–Riemann equations in passing (as had d'Alembert and Euler) as general useful property of analytic functions, rather than fundamental feature of the theory

Complex numbers in the *Cours d'analyse* (1821)



176 COURS D'ANALYSE.

toute expression symbolique de la forme

$$\alpha + \mathcal{6}\sqrt{-1},$$

α, $\mathcal{6}$ désignant deux quantités réelles ; et l'on dit que deux expressions imaginaires

$$\alpha + \mathcal{6}\sqrt{-1}, \quad \gamma + \delta\sqrt{-1}$$

sont *égales* entre elles, lorsqu'il y a égalité de part et d'autre, 1.° entre les parties réelles α et γ, 2.° entre les coefficiens de $\sqrt{-1}$, savoir, $\mathcal{6}$ et δ. L'égalité de deux expressions imaginaires s'indique, comme celle de deux quantités réelles, par le signe $=$; et il en résulte ce qu'on appelle une *équation imaginaire*. Cela posé, toute équation imaginaire n'est que la représentation symbolique de deux équations entre quantités réelles. Par exemple, l'équation symbolique

$$\alpha + \mathcal{6}\sqrt{-1} = \gamma + \delta\sqrt{-1}$$

équivaut seule aux deux équations réelles

$$\alpha = \gamma, \quad \mathcal{6} = \delta.$$

Lorsque, dans l'expression imaginaire

$$\alpha + \mathcal{6}\sqrt{-1},$$

le coefficient $\mathcal{6}$ de $\sqrt{-1}$ s'évanouit, le terme $\mathcal{6}\sqrt{-1}$ est censé réduit à zéro, et l'expression elle-même à la quantité réelle α. En vertu de cette convention, les expressions imaginaires comprennent, comme cas particuliers, les quantités réelles.

Les expressions imaginaires peuvent être sou-

Defined as "symbolic expressions" $a + b\sqrt{-1}$

55-page development of formal definitions and properties

Consideration of multi-functions — which are the most natural branches to take?

Sought to extend ideas for real functions to the complex case, particularly those relating to power series and convergence

Cauchy's second 'Mémoire' (1825)

'Mémoire sur les intégrales définies, prises entre des limites imaginaires'

Direct adaptation of definition of real integral to the complex case:

$$\int_{x_0+y_0\sqrt{-1}}^{X+Y\sqrt{-1}} f(z)dz$$

is the limit (or one of the limits) of a sum of products of the form

$$\sum (x_{i-1} + y_{i-1}\sqrt{-1}) f(x_{i-1} + y_{i-1}\sqrt{-1}).$$

NB. No explicit definition of a function of a complex variable; tacit assumption of differentiability, hence that the Cauchy–Riemann equations hold.

Contour integration

In any domain where the function does not become infinite, the value of a complex integral is independent of the path along which it is taken.

Cauchy: consider two different paths within the rectangle (x_0, y_0), (X, Y) such that the function $f(x + y\sqrt{-1})$ does not become infinite for values of x, y lying within the domain enclosed by the paths. Then the value of the integral $\int_{x_0+y_0\sqrt{-1}}^{X+Y\sqrt{-1}} f(z)dz$ is independent of the path taken.

Really a theorem about real functions in the plane?

(Gauss had discovered this in 1811, alongside a similar definition of a complex integral, but did not publish.)

Contour integration

For the case where $f(x + y\sqrt{-1})$ becomes infinite at the point $x = a, y = b$, Cauchy considered the limit

$$\mathrm{f} := \lim_{\substack{x \to a \\ y \to b}} \left(x - a + (y - b)\sqrt{-1}\right) f\left(x + y\sqrt{-1}\right),$$

and determined that the difference between the integrals of f along different paths that are infinitely close to each other as well as to (a, b) is $2\pi \mathrm{f}\sqrt{-1}$.

With a natural extension of this result for multiple and/or higher-order singularities, this became an ancestor of Cauchy's residue theorem — developed as part of Cauchy's calculus of residues in a paper of 1826 ('Sur un nouveau genre de calcul').

Taylor's Theorem for complex analytic functions

In *Cours d'analyse* (1821), Cauchy had considered the notion of radius of convergence for both real and imaginary power series.

1831: a complex function has a convergent power series if it is "finite and continuous"

Continued to refine the conditions for the theorem over many years.

Cauchy's language is not always satisfactory to modern eyes, but was considerably more rigorous than that of most of his contemporaries.

1841: extension to negative powers — Laurent's Theorem.

Cauchy's complex analysis

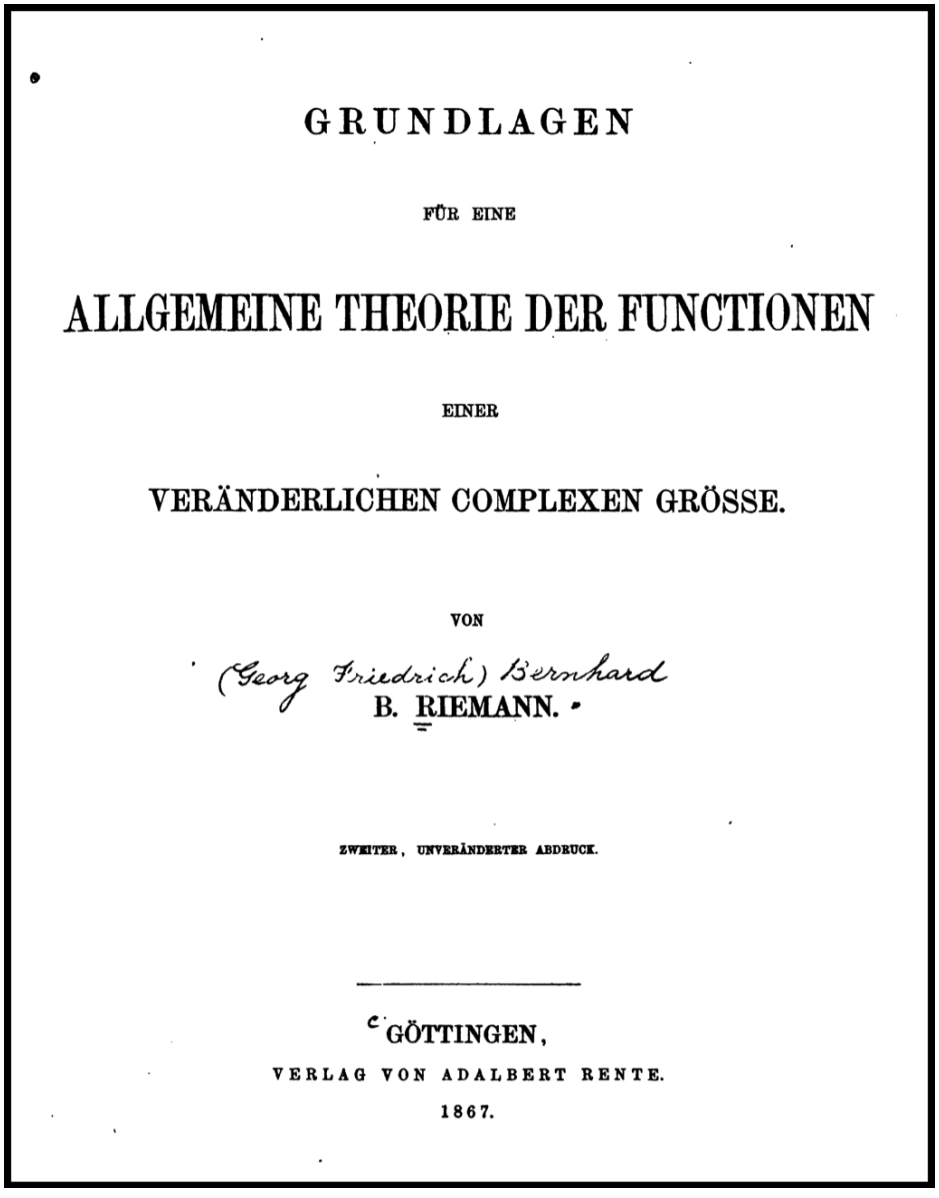
Cauchy's ideas concerning complex functions developed over many years. In the early stages

- did he appreciate the fundamental nature of the concepts and results that he was using and deriving?
- did he recognise the subtleties of working with complex numbers rather than simply with pairs of real numbers?

Have historians of mathematics read too much into the earlier work on the basis of what came later?

Point to note: Cauchy may be credited with many of the fundamental ideas of complex analysis, but this does not mean that they appeared fully-formed.

Riemann on complex analysis



GRUNDLAGEN

FÜR EINE

ALLGEMEINE THEORIE DER FUNCTIONEN

EINER

VERÄNDERLICHEN COMPLEXEN GRÖSSE.

VON

(Georg Friedrich) Bernhard
B. RIEMANN.

ZWEITER, UNVERÄNDERTER ABDRUCK.

GÖTTINGEN,
VERLAG VON ADALBERT RENTE.
1867.

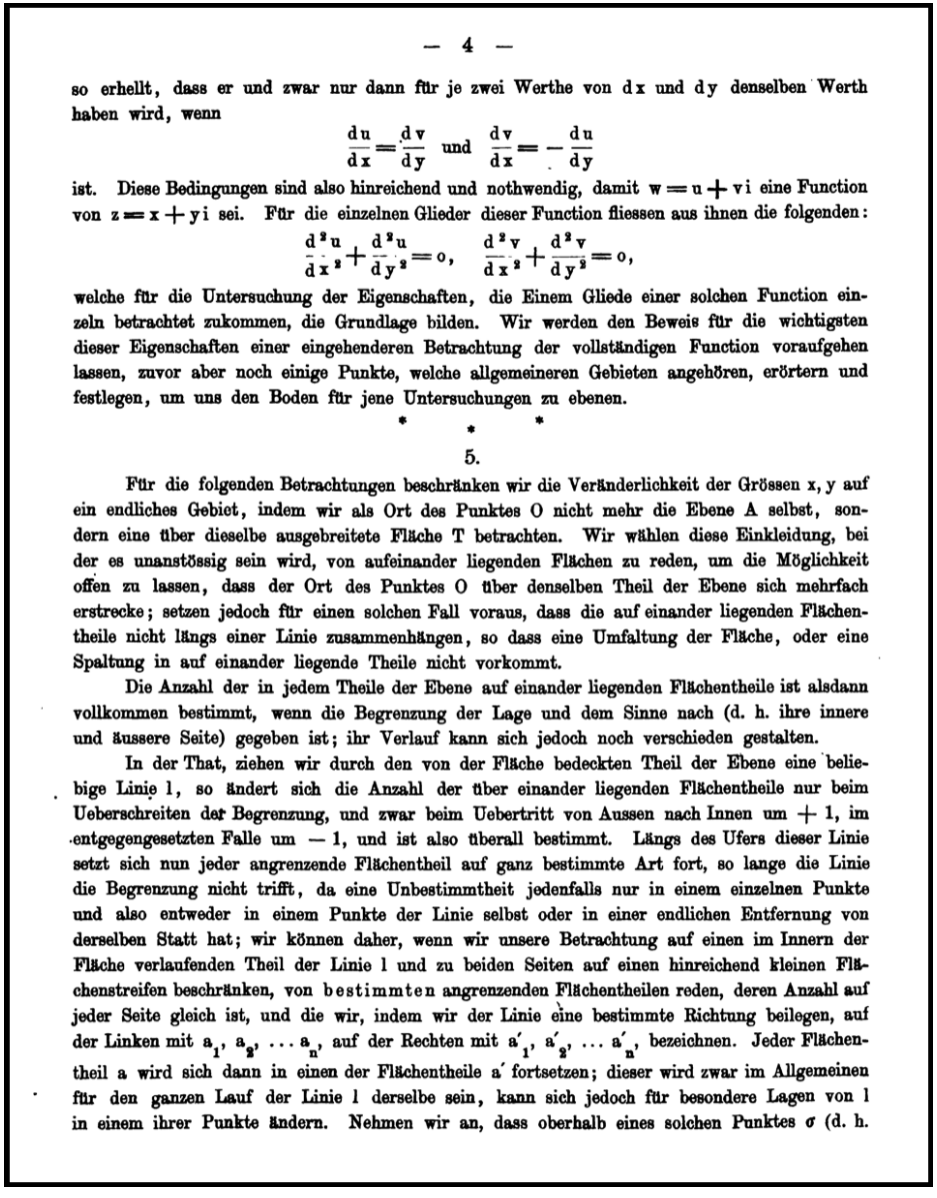
Doctoral dissertation: *Foundations for a General Theory of Functions of a Variable Complex Quantity* (1851)

Started from the idea that a complex variable should be treated as a single quantity z

"The complex variable w is called a function of another complex variable z when its variation is such that the value of the derivative $\frac{dw}{dz}$ is independent of the value of dz"

That is: $\lim_{\delta \to 0} \frac{f(z+\delta)-f(z)}{\delta}$ exists

Riemann on complex analysis



— 4 —

so erhellt, dass er und zwar nur dann für je zwei Werthe von dx und dy denselben Werth haben wird, wenn

$$\frac{du}{dx}=\frac{dv}{dy} \quad \text{und} \quad \frac{dv}{dx}=-\frac{du}{dy}$$

ist. Diese Bedingungen sind also hinreichend und nothwendig, damit $w=u+vi$ eine Function von $z=x+yi$ sei. Für die einzelnen Glieder dieser Function fliessen aus ihnen die folgenden:

$$\frac{d^2u}{dx^2}+\frac{d^2u}{dy^2}=0, \quad \frac{d^2v}{dx^2}+\frac{d^2v}{dy^2}=0,$$

welche für die Untersuchung der Eigenschaften, die Einem Gliede einer solchen Function einzeln betrachtet zukommen, die Grundlage bilden. Wir werden den Beweis für die wichtigsten dieser Eigenschaften einer eingehenderen Betrachtung der vollständigen Function voraufgehen lassen, zuvor aber noch einige Punkte, welche allgemeineren Gebieten angehören, erörtern und festlegen, um uns den Boden für jene Untersuchungen zu ebenen.

* * *

5.

Für die folgenden Betrachtungen beschränken wir die Veränderlichkeit der Grössen x, y auf ein endliches Gebiet, indem wir als Ort des Punktes O nicht mehr die Ebene A selbst, sondern eine über dieselbe ausgebreitete Fläche T betrachten. Wir wählen diese Einkleidung, bei der es unanstössig sein wird, von aufeinander liegenden Flächen zu reden, um die Möglichkeit offen zu lassen, dass der Ort des Punktes O über denselben Theil der Ebene sich mehrfach erstrecke; setzen jedoch für einen solchen Fall voraus, dass die auf einander liegenden Flächentheile nicht längs einer Linie zusammenhängen, so dass eine Umfaltung der Fläche, oder eine Spaltung in auf einander liegende Theile nicht vorkommt.

Die Anzahl der in jedem Theile der Ebene auf einander liegenden Flächentheile ist alsdann vollkommen bestimmt, wenn die Begrenzung der Lage und dem Sinne nach (d. h. ihre innere und äussere Seite) gegeben ist; ihr Verlauf kann sich jedoch noch verschieden gestalten.

In der That, ziehen wir durch den von der Fläche bedeckten Theil der Ebene eine beliebige Linie l, so ändert sich die Anzahl der über einander liegenden Flächentheile nur beim Ueberschreiten der Begrenzung, und zwar beim Uebertritt von Aussen nach Innen um + 1, im entgegengesetzten Falle um − 1, und ist also überall bestimmt. Längs des Ufers dieser Linie setzt sich nun jeder angrenzende Flächentheil auf ganz bestimmte Art fort, so lange die Linie die Begrenzung nicht trifft, da eine Unbestimmtheit jedenfalls nur in einem einzelnen Punkte und also entweder in einem Punkte der Linie selbst oder in einer endlichen Entfernung von derselben Statt hat; wir können daher, wenn wir unsere Betrachtung auf einen im Innern der Fläche verlaufenden Theil der Linie l und zu beiden Seiten auf einen hinreichend kleinen Flächenstreifen beschränken, von bestimmten angrenzenden Flächentheilen reden, deren Anzahl auf jeder Seite gleich ist, und die wir, indem wir der Linie eine bestimmte Richtung beilegen, auf der Linken mit $a_1, a_2, \ldots a_n$, auf der Rechten mit $a'_1, a'_2, \ldots a'_n$, bezeichnen. Jeder Flächentheil a wird sich dann in einen der Flächentheile a′ fortsetzen; dieser wird zwar im Allgemeinen für den ganzen Lauf der Linie l derselbe sein, kann sich jedoch für besondere Lagen von l in einem ihrer Punkte ändern. Nehmen wir an, dass oberhalb eines solchen Punktes σ (d. h.

Cauchy–Riemann equations now taken as fundamental to the theory

Other key concepts appear explicitly:

- harmonic functions;
- conformality (a complex function preserves angles wherever its derivative does not vanish);
- ...

Early impact limited by abstraction and restricted publication

The word ‘analytic’

The words analysis, analytic have had many meanings:

Classical:	a method of investigating a problem, the opposite of synthesis
c. 1600:	algebra became known as the ‘analytic art’ or just ‘analysis’, using finite equations
1669:	Newton introduced ‘analysis with infinite equations’, that is, infinite series
1748:	Euler wrote on the analysis of infinitely large and infinitely small quantities
1790–1840:	in sections of journals, the Académie des Sciences, etc., Analyse could mean ‘pure mathematics’ though with a bias to algebra, calculus, etc.; compare Géométrie also meaning ‘pure mathematics’, but with (perhaps) spatial bias
1821:	Cauchy’s cours d’analyse shows similarities with our analysis courses today

What *is* an analytic function?

Lagrange, 1797: function is analytic if it has a power-series expansion

Cauchy's point of departure, 1814–1831: treated complex functions that are continuous and satisfy the Cauchy–Riemann equations (always true for analytic functions in the sense of Lagrange), but used no special terminology

Riemann, 1851: switched focus to complex functions for which $\lim_{h\to 0} \frac{f(z+h)-f(z)}{h}$ exists in the region of interest

Weierstrass, 1860s: applied Lagrange's term analytic to Riemann's conception of function

Oxford, 2022: we follow Riemann and Weierstrass, by using the words holomorphic, meromorphic, etc. as variants of analytic, with slightly different meanings

BO1 History of Mathematics
Lecture XIV
Linear algebra

MT 2022 Week 7

Summary

- Linear equations
- Determinants
- Eigenvalues
- Matrices
- Vector spaces

Difficulties in the historical study of linear algebra

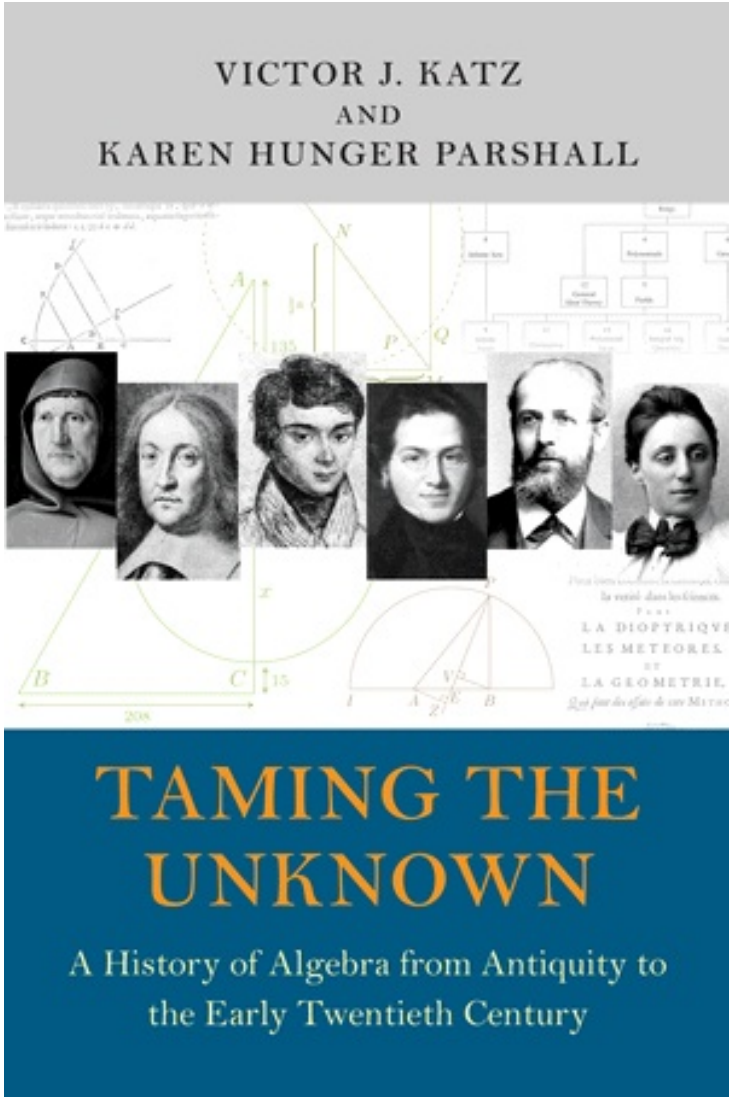
> *Linear algebra may be mathematically simple but its history is more complicated than any other topic in this book. . . . [Its development is] a very tangled tale.*

(*Mathematics Emerging*, p. 548)

- linear algebra is elementary but its manifestations are many and sophisticated
- there are hardly any obvious starting points
- theory often lagged behind practice
- practice sometimes lagged behind theory
- 19th-century reliance on theory of quadratic and bilinear forms (e.g., $ax^2 + 2bxy + cy^2$) — unfamiliar to students now

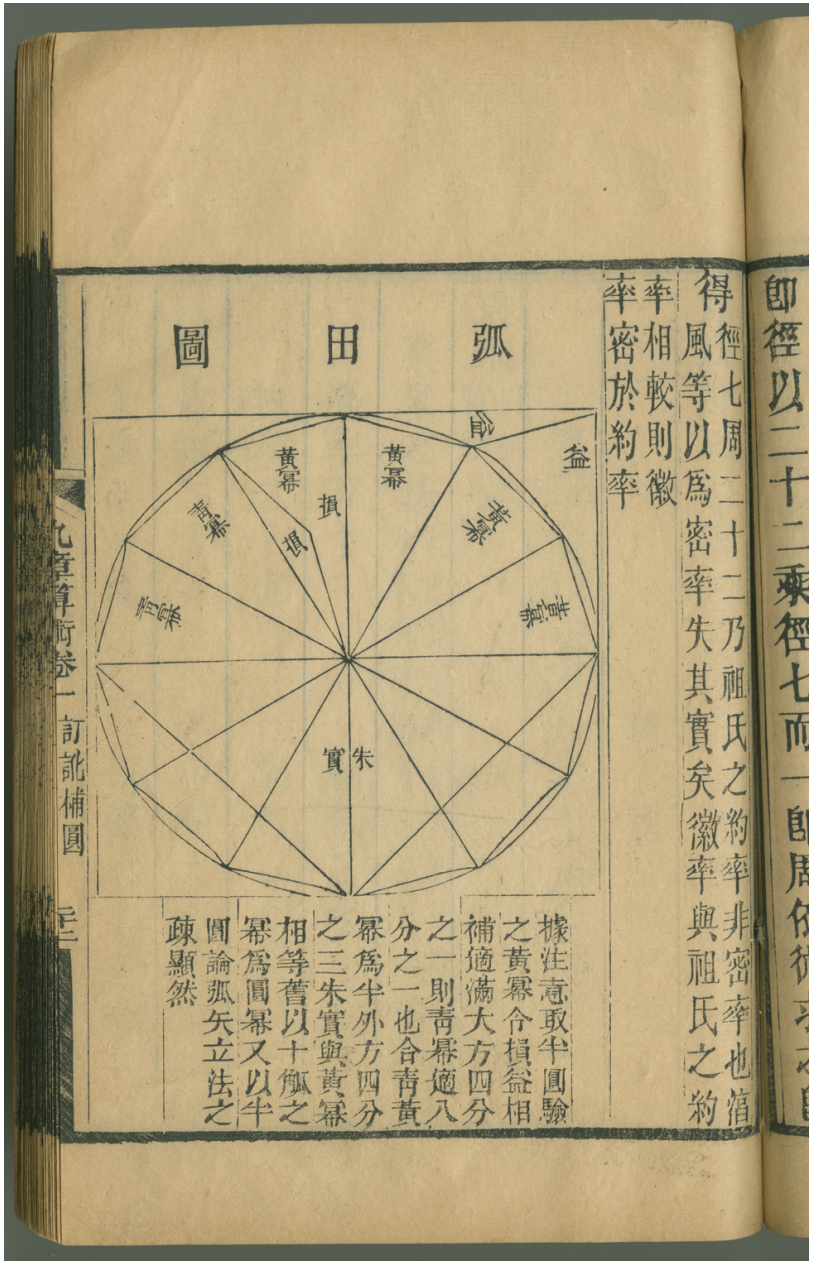
Warning: matrices (etc.) are primary in modern teaching, determinants secondary. For about 200 years until 1940 (or thereabouts) the reverse was the case: determinants came first.

On the history of linear algebra


VICTOR J. KATZ
AND
KAREN HUNGER PARSHALL
TAMING THE UNKNOWN
A History of Algebra from Antiquity to the Early Twentieth Century

(Princeton University Press, 2014)

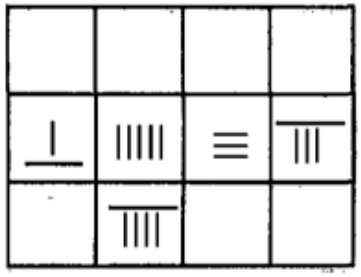
Jiǔzhāng Suànshù (China, c. 150 BC)

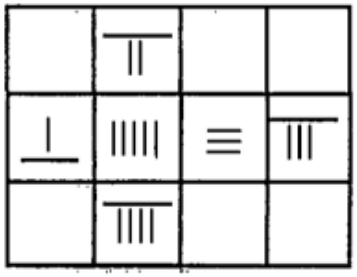


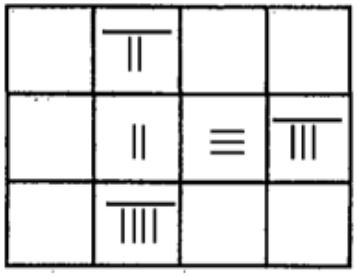
Nine chapters of the mathematical art 九章算術 (from a 16th-century edition, derived from a 3rd-century commentary by Liu Hui 劉徽)

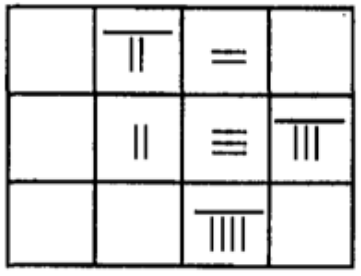
Content: calculation of areas ($\pi \approx 3.14159$), rates of exchange, computation with fractions, proportion, extraction of square and cube roots, calculation of volumes, systems of linear equations, Pythagoras' Theorem, ...

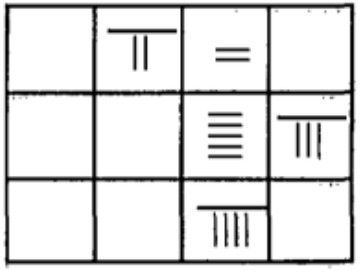
Chinese calculation

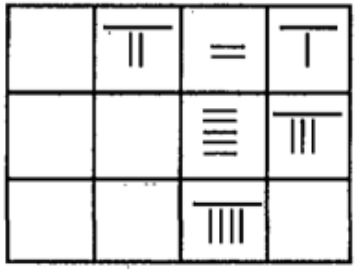


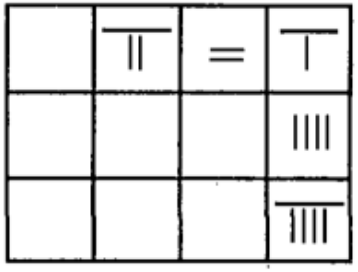












```
   726
9 ⟌6538
  63
  ──
   23
   18
   ──
    58
    54
    ──
     4
```

Base 10 system of rods on counting board: red for positive, black for negative

Early linear equations in China

Chapter 7: solution of pairs of equations in two unknowns by the method of false position

Chapter 8: solution of systems of n equations in n unknowns for $n \leq 5$

There are three types of grain

3 bundles of the first, 2 of the second, and 1 of the third contain 39 measures
2 of the first, 3 of the second, and 1 of the third contain 34
1 of the first, 2 of the second, and 3 of the third contain 26

How many measures in a bundle of each type?

Solved on a counting board by Gaussian elimination, known here as 'fāngchéng' 方程

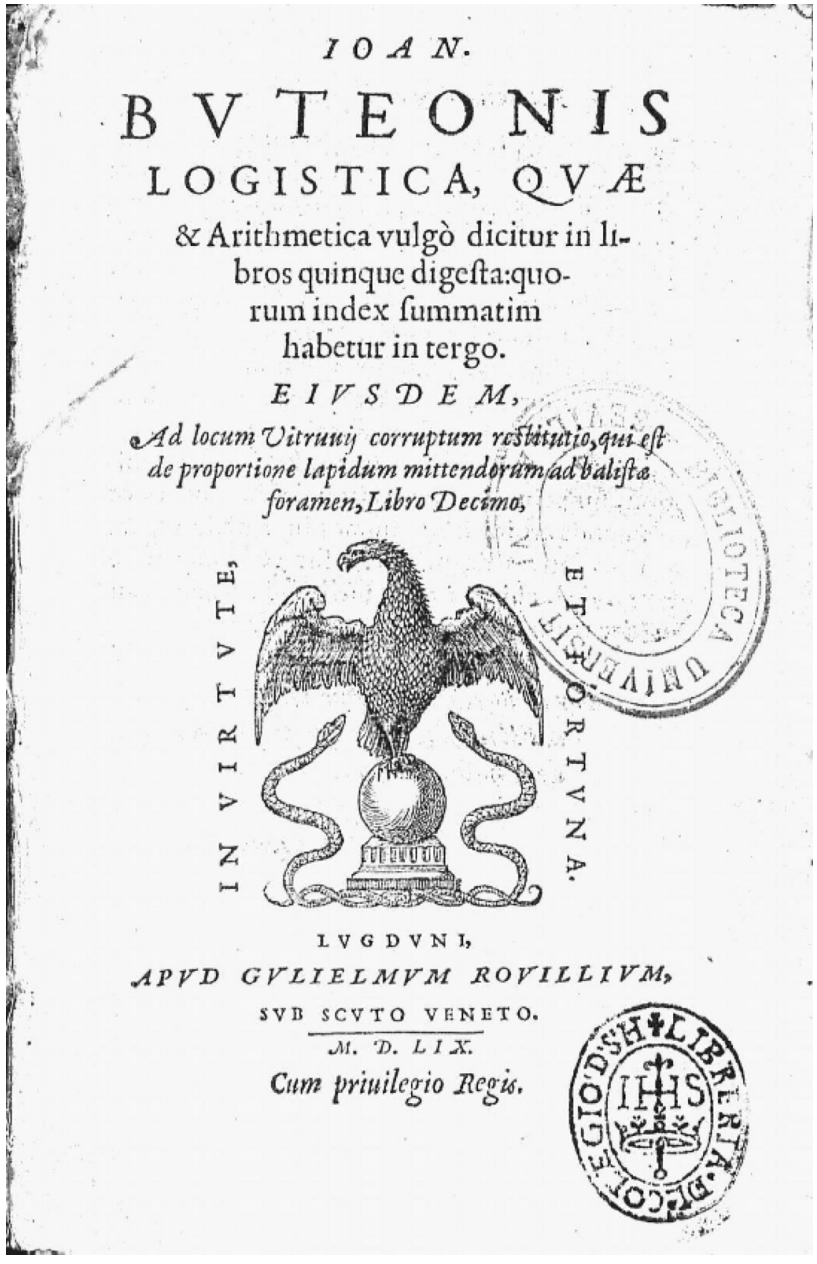
Early linear equations in China

There are five families which share a well. 2 of A's ropes are short of the well's depth by 1 of B's ropes. 3 of B's ropes are short of the depth by 1 of C's ropes. 4 of C's ropes are short by 1 of D's ropes. 5 of D's ropes are short by 1 of E's ropes. 6 of E's ropes are short by 1 of A's ropes. Find the depth of the well and the length of each rope.

Five equations in six unknowns, so indeterminate

Liu Hui: we can only give a solution in terms of proportions of the lengths

Early linear equations in Europe



IOAN.

BVTEONIS

LOGISTICA, QVÆ

& Arithmetica vulgò dicitur in libros quinque digesta: quorum index summatim habetur in tergo.

EIVSDEM,

Ad locum Vitruuij corruptum restitutio, qui est de proportione lapidum mittendorum ad balistæ foramen, Libro Decimo,

IN VIRTVTE, ET FORTVNA.

LVGDVNI,

APVD GVLIELMVM ROVILLIVM,

SVB SCVTO VENETO.

M. D. LIX.

Cum priuilegio Regis.

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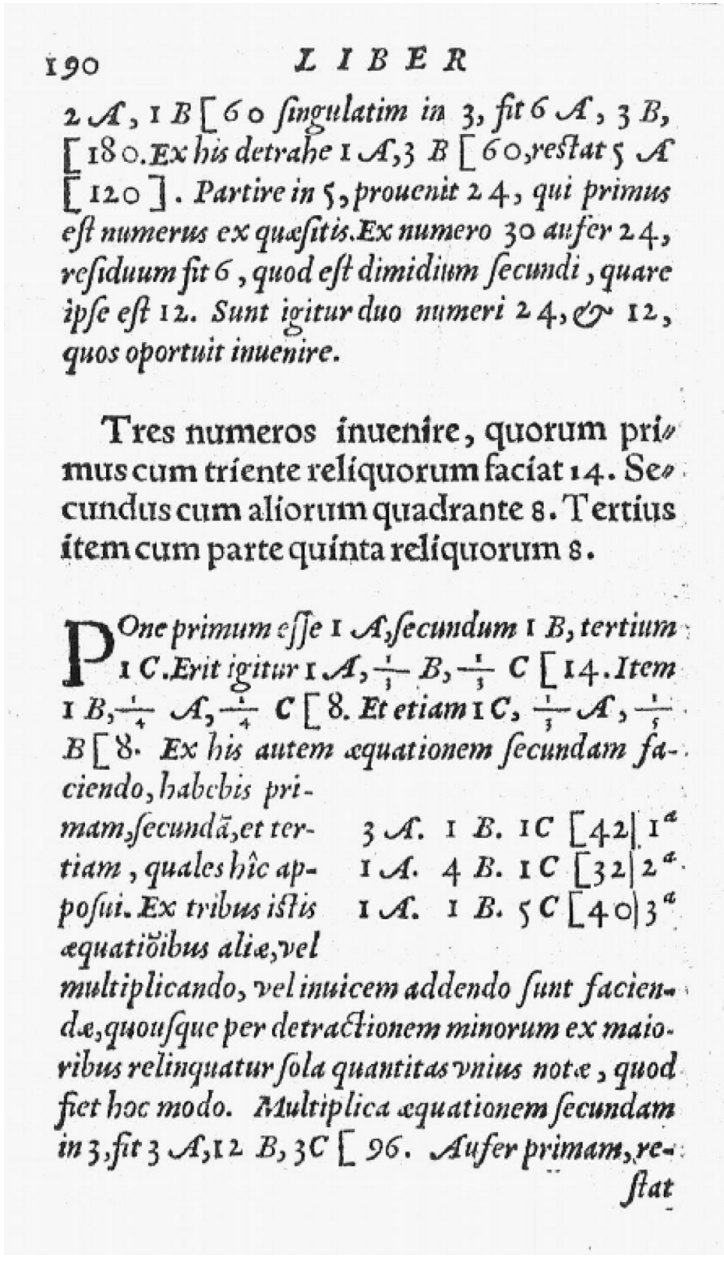
Jean Borrel [Ioannes Buteus] *Logistica, quæ et Arithmetica vulgo dicitur in libros quinque digesta* (*Logistic, also known as Arithmetic, digested in five books*), 1559

Linear equations in Borrel's *Logistica*



190 LIBER

2 A, 1 B [60 singulatim in 3, fit 6 A, 3 B, [180. Ex his detrahe 1 A, 3 B [60, restat 5 A [120]. Partire in 5, prouenit 24, qui primus est numerus ex quæsitis. Ex numero 30 aufer 24, residuum fit 6, quod est dimidium secundi, quare ipse est 12. Sunt igitur duo numeri 24, & 12, quos oportuit inuenire.

Tres numeros inuenire, quorum primus cum triente reliquorum faciat 14. Secundus cum aliorum quadrante 8. Tertius item cum parte quinta reliquorum 8.

PONE primum esse 1 A, secundum 1 B, tertium 1 C. Erit igitur 1 A, $\frac{1}{3}$ B, $\frac{1}{3}$ C [14. Item 1 B, $\frac{1}{4}$ A, $\frac{1}{4}$ C [8. Et etiam 1 C, $\frac{1}{3}$ A, $\frac{1}{5}$ B [8. Ex his autem æquationem secundam faciendo, habebis primam, secundã, et tertiam, quales hîc apposui. Ex tribus istis æquatiõibus aliæ, vel multiplicando, vel inuicem addendo sunt faciendæ, quousque per detractionem minorum ex maioribus relinquatur sola quantitas vnius notæ, quod fiet hoc modo. Multiplica æquationem secundam in 3, fit 3 A, 12 B, 3 C [96. Aufer primam, re-

3 A.	1 B.	1 C	[42	1ª
1 A.	4 B.	1 C	[32	2ª
1 A.	1 B.	5 C	[40	3ª

stat

To find three numbers, of which the first with a third of the rest makes 14. The second with a quarter of the rest makes 8. Likewise the third with a fifth part of the rest makes 8.

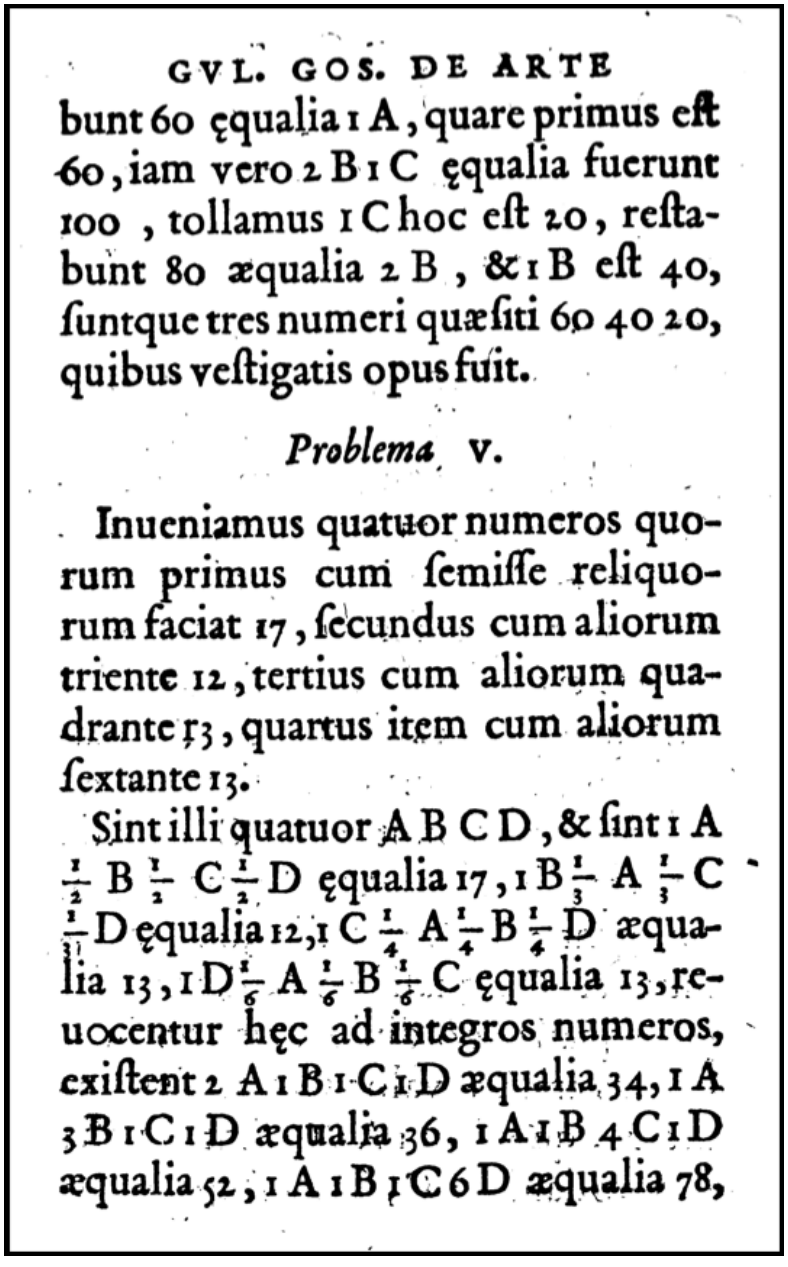
Put the first to be 1*A*, *the second* 1*B*, *the third* 1*C*. . . .

[Derives a system of equations with '.' for addition and '[' for equality.]

Multiply by 3, by 4 and by 5 respectively, etc.

(See *Mathematics emerging*, §17.1.1.)

More unknowns



GVL. GOS. DE ARTE

bunt 60 æqualia 1 A, quare primus eſt 60, iam vero 2 B 1 C æqualia fuerunt 100, tollamus 1 C hoc eſt 20, reſtabunt 80 æqualia 2 B, & 1 B eſt 40, ſuntque tres numeri quæſiti 60 40 20, quibus veſtigatis opus fuit.

Problema V.

Inueniamus quatuor numeros quorum primus cum ſemiſſe reliquorum faciat 17, ſecundus cum aliorum triente 12, tertius cum aliorum quadrante 13, quartus item cum aliorum ſextante 13.

Sint illi quatuor A B C D, & ſint 1 A $\frac{1}{2}$ B $\frac{1}{2}$ C $\frac{1}{2}$ D æqualia 17, 1 B $\frac{1}{3}$ A $\frac{1}{3}$ C $\frac{1}{3}$ D æqualia 12, 1 C $\frac{1}{4}$ A $\frac{1}{4}$ B $\frac{1}{4}$ D æqualia 13, 1 D $\frac{1}{6}$ A $\frac{1}{6}$ B $\frac{1}{6}$ C æqualia 13, reuocentur hæc ad integros numeros, exiſtent 2 A 1 B 1 C 1 D æqualia 34, 1 A 3 B 1 C 1 D æqualia 36, 1 A 1 B 4 C 1 D æqualia 52, 1 A 1 B 1 C 6 D æqualia 78,

Guillaume Gosselin, *De arte magna seu de occulta parte numerorum quae et Algebra et Almucabala vulgo dicitur* (*On the great art or the hidden part of numbers commonly called Algebra and Almucabala*), 1577

$$1A + \frac{1}{2}B + \frac{1}{2}C + \frac{1}{2}D = 17$$
$$1B + \frac{1}{3}A + \frac{1}{3}C + \frac{1}{3}D = 12$$
$$1C + \frac{1}{4}A + \frac{1}{4}B + \frac{1}{4}D = 13$$
$$1D + \frac{1}{6}A + \frac{1}{6}B + \frac{1}{6}C = 13$$

A 17th-century example

After reading Gosselin . . .
John Pell to Sir Charles Cavendish (1646):

Exemplum . . . satis determinatis

$$3a - 4b + 5c = 2$$
$$5a + 3b - 2c = 58$$
$$7a - 5b + 4c = 14$$

(Solved via Pell's 'three-column method')

Exemplum . . . non satis determinatis

$$5a + 3b - 2c = 24$$
$$-2a + 4b + 3c = 5$$

($a, b, c > 0$; found bounds for the possible values: e.g., $a < 15\frac{9}{11}$)

Linear equations — systematic practical methods

Gaussian elimination:

- *The nine chapters of the mathematical art*, China (c. 150 BC)
- Colin Maclaurin, *A treatise of algebra* (1748), §§82–85

Maclaurin on Gaussian elimination

A

TREATISE

OF

ALGEBRA,

IN

THREE PARTS.

CONTAINING

I. *The Fundamental Rules and Operations.*

II. *The Compoſition and Reſolution of Equations of all Degrees; and the different Affections of their Roots.*

III. *The Application of* Algebra *and* Geometry *to each other.*

To which is added an

APPENDIX,

Concerning the general Properties of GEOMETRICAL LINES.

By *COLIN MACLAURIN*, M. A.
Late PROFESSOR of MATHEMATICS in the University of *Edinburgh*, and *Fellow* of the *Royal Society*.

LONDON:
Printed for A. MILLAR, and J. NOURSE, oppoſite to *Catherine-Street*, in the *Strand*.
M.DCC.XLVIII.

$$\begin{cases} x : y :: a : b \\ x^2 - y^2 = d \end{cases}$$

$$x = \frac{ay}{b} \text{ and } x^2 = \frac{a^2y^2}{b^2}$$

but $x^2 = d + y^2$

whence $d + y^2 = \frac{a^2y^2}{b^2}$

and $a^2y^2 - b^2y^2 = db^2$

$$y^2 = \frac{db^2}{a^2 - b^2}$$

$$y = \sqrt[3]{\frac{db^2}{a^2 - b^2}}$$

and $x = \sqrt[3]{\frac{da^2}{a^2 - b^2}}$.

DIRECTION V.

§ 82. "*If there are three unknown Quantities, there muſt be three Equations in order to determine them, by comparing which you may, in all Caſes, find two Equations involving only two unknown Quantities; and then, by* Direct. 3d, *from theſe two you may deduce an Equation involving only one unknown Quantity; which may be reſolved by the Rules of the laſt Chapter.*"

From 3 Equations involving any three unknown Quantities, x, y, and z, to deduce two Equations involving only two unknown Quantities, the following Rule will always ſerve.

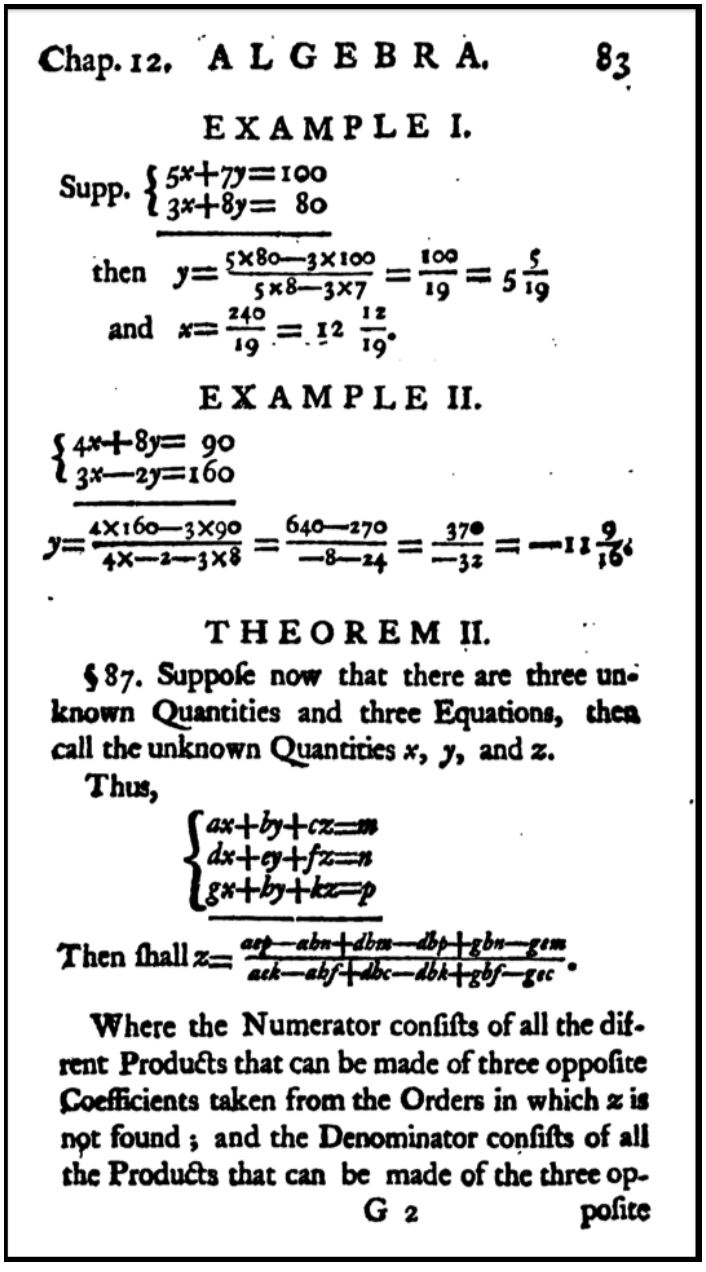
RULE.

Linear equations — systematic practical methods

Gaussian elimination:

- *The nine chapters of the mathematical art*, China (c. 150 BC)
- Colin Maclaurin, *A treatise of algebra* (1748), §§82–85
- C. F. Gauss: calculation of asteroid orbits (1810)
- from surveying, e.g., Wilhelm Jordan, *Handbuch der Vermessungskunde*, 3rd edition (1888)

Maclaurin and linear equations



Chap. 12. ALGEBRA. 83

EXAMPLE I.

Supp. $\begin{cases} 5x+7y=100 \\ 3x+8y=80 \end{cases}$

then $y=\frac{5\times80-3\times100}{5\times8-3\times7}=\frac{100}{19}=5\frac{5}{19}$

and $x=\frac{240}{19}=12\frac{12}{19}$.

EXAMPLE II.

$\begin{cases} 4x+8y=90 \\ 3x-2y=160 \end{cases}$

$y=\frac{4\times160-3\times90}{4\times-2-3\times8}=\frac{640-270}{-8-24}=\frac{370}{-32}=-11\frac{9}{16}$

THEOREM II.

§87. Suppose now that there are three unknown Quantities and three Equations, then call the unknown Quantities x, y, and z.

Thus,

$$\begin{cases} ax+by+cz=m \\ dx+ey+fz=n \\ gx+hy+kz=p \end{cases}$$

Then shall $z=\frac{aep-abn+dbm-dhp+ghn-gem}{aek-ahf+dhc-dbk+gbf-gec}$.

Where the Numerator consists of all the different Products that can be made of three opposite Coefficients taken from the Orders in which z is not found; and the Denominator consists of all the Products that can be made of the three opposite

G 2

Colin Maclaurin, *A treatise of algebra*, 1748, p. 83

Three equations in three unknowns solved using a 'determinant-like' quantity

Chap. 13. ALGEBRA. 85

If four Equations are given, involving four unknown Quantities, their Values may be found much after the same Manner, by taking all the Products that can be made of four opposite Coefficients, and always prefixing contrary Signs to those that involve the Products of two opposite Coefficients.

Notational difficulties — we run out of letters!

Determinants

Colin Maclaurin, *A treatise of algebra*, 1748, Ch. XII, pp. 81–85

Determinants

Leibniz, unpublished works, 1680s/1690s.

Colin Maclaurin, *A treatise of algebra*, 1748, Ch. XII, pp. 81–85

Leibniz and determinants

At least as early as June 1678, Leibniz devised a new notation for coefficients, writing

$$10 + 11x + 12y = 0,$$
$$20 + 21x + 22y = 0$$

for what we would write as

$$a_{10} + a_{11}x + a_{12}y = 0,$$
$$a_{20} + a_{21}x + a_{22}y = 0.$$

Leibniz used this notation to formulate general results on the solvability of systems of equations in terms of a determinant-like quantity (a sum of signed products of coefficients) — but these were not published during his lifetime

Determinants

Leibniz, unpublished works, 1680s/1690s.

Colin Maclaurin, *A treatise of algebra*, 1748, Ch. XII, pp. 81–85

Determinants

Leibniz, unpublished works, 1680s/1690s.

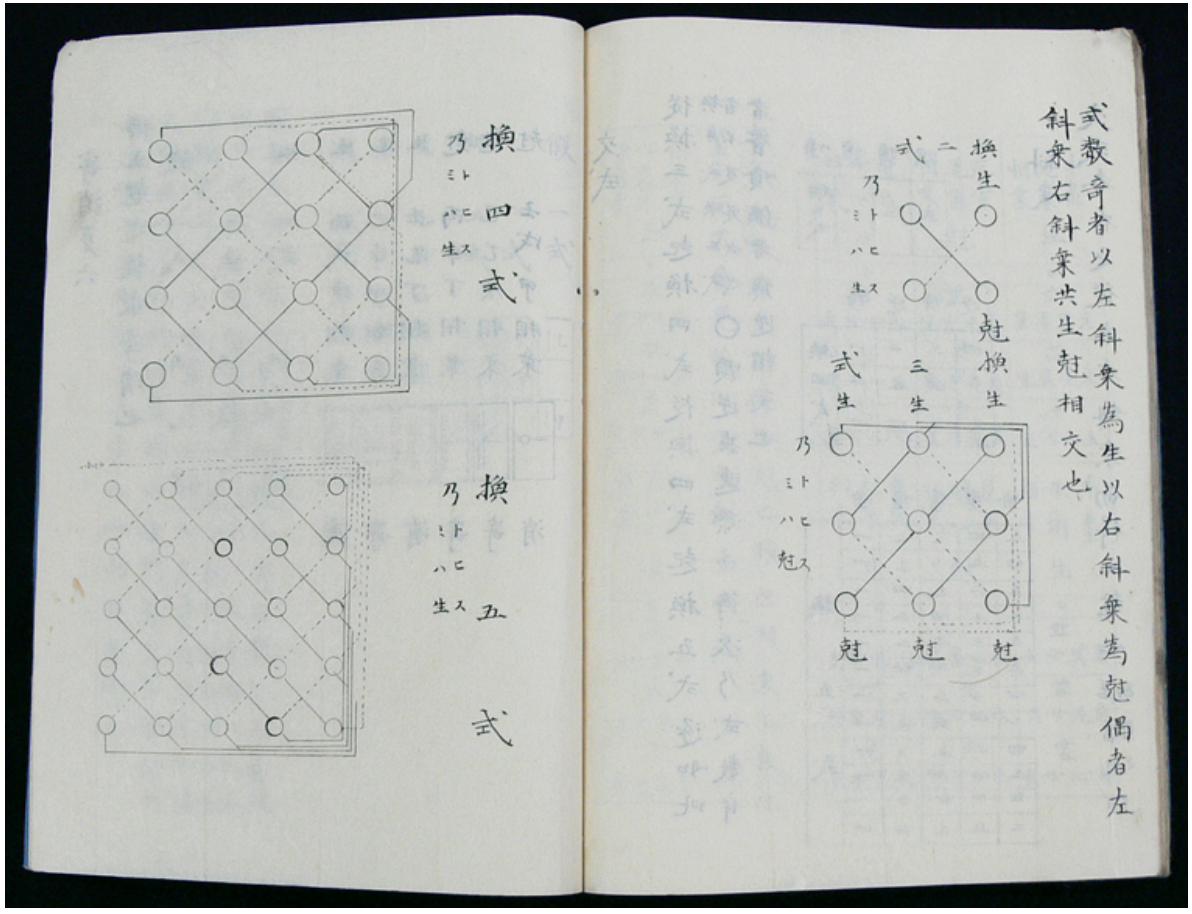
Seki Takakazu, *Kai-fukudai-no-hō* 解伏題之法, 1683

Colin Maclaurin, *A treatise of algebra*, 1748, Ch. XII, pp. 81–85

Seki and determinants

Seki Takakazu, *Kai-fukudai-no-hō* 解伏題之法 (*Method for Solving Concealed Problems*), 1683

Arranged coefficients of systems of equations in a grid, and gave schematics for construction of determinants (dotted lines indicate positive products, and solid lines negative)



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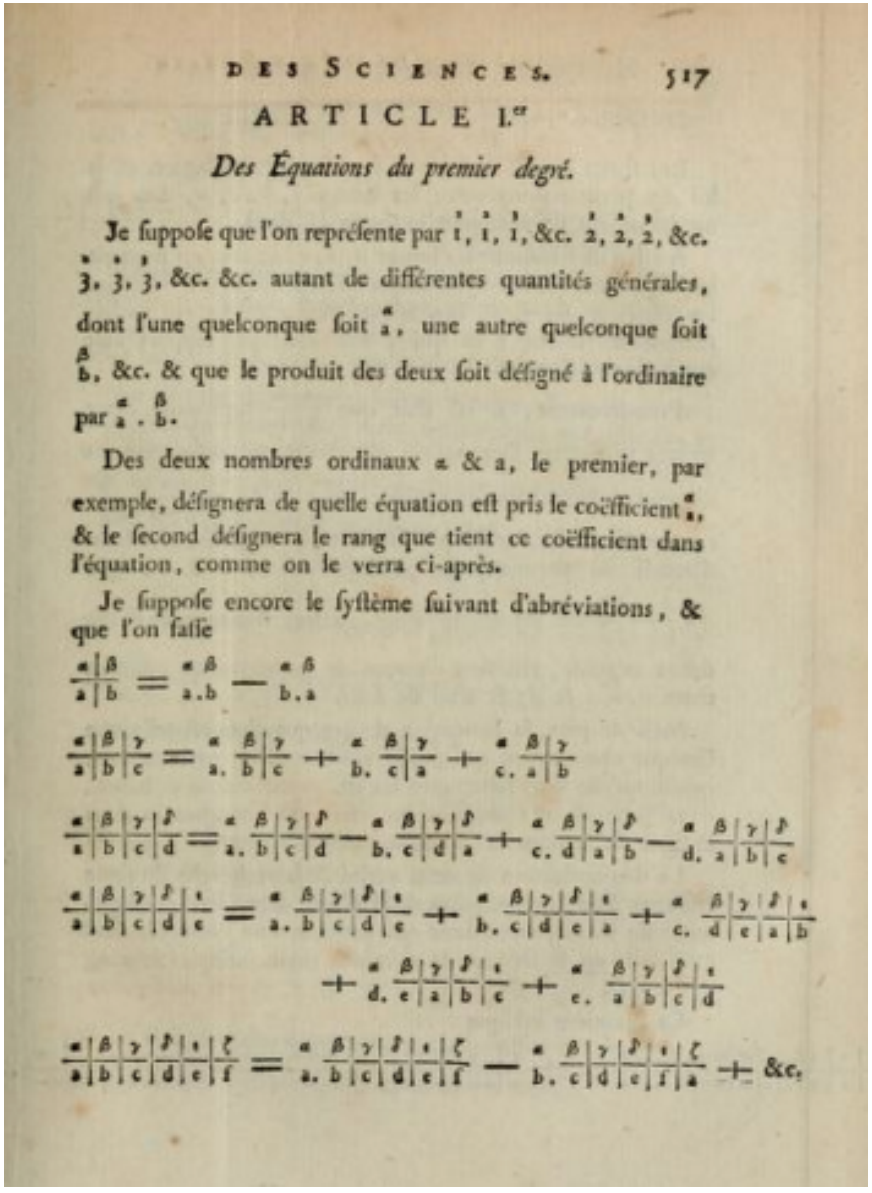
Leibniz, unpublished works, 1680s/1690s.

Seki Takakazu, *Kai-fukudai-no-hō* 解伏題之法, 1683

Colin Maclaurin, *A treatise of algebra*, 1748, Ch. XII, pp. 81–85

Vandermonde, 'Mémoire sur l'élimination', *Mémoires de l'Académie des sciences*, 1772: a recursive description of determinants of any size (but without a name and in an uncongenial notation — see *Mathematics emerging*, §17.1.3)

Vandermonde on elimination



DES SCIENCES. 517

ARTICLE I.er

Des Équations du premier degré.

Je ſuppoſe que l'on repréſente par $\overset{\prime}{1}, \overset{\prime\prime}{1}, \overset{\prime\prime\prime}{1}$, &c. $\overset{\prime}{2}, \overset{\prime\prime}{2}, \overset{\prime\prime\prime}{2}$, &c. $\overset{\prime}{3}, \overset{\prime\prime}{3}, \overset{\prime\prime\prime}{3}$, &c. &c. autant de différentes quantités générales, dont l'une quelconque ſoit $\overset{\alpha}{a}$, une autre quelconque ſoit $\overset{\beta}{b}$, &c. & que le produit des deux ſoit déſigné à l'ordinaire par $\overset{\alpha}{a} . \overset{\beta}{b}$.

Des deux nombres ordinaux α & a, le premier, par exemple, déſignera de quelle équation eſt pris le coëfficient $\overset{\alpha}{a}$, & le ſecond déſignera le rang que tient ce coëfficient dans l'équation, comme on le verra ci-après.

Je ſuppoſe encore le ſyſtême ſuivant d'abréviations, & que l'on faſſe

$$\frac{\alpha|\beta}{a|b} = \overset{\alpha}{a}.\overset{\beta}{b} - \overset{\alpha}{b}.\overset{\beta}{a}$$

$$\frac{\alpha|\beta|\gamma}{a|b|c} = \overset{\alpha}{a}.\frac{\beta|\gamma}{b|c} + \overset{\alpha}{b}.\frac{\beta|\gamma}{c|a} + \overset{\alpha}{c}.\frac{\beta|\gamma}{a|b}$$

$$\frac{\alpha|\beta|\gamma|\delta}{a|b|c|d} = \overset{\alpha}{a}.\frac{\beta|\gamma|\delta}{b|c|d} - \overset{\alpha}{b}.\frac{\beta|\gamma|\delta}{c|d|a} + \overset{\alpha}{c}.\frac{\beta|\gamma|\delta}{d|a|b} - \overset{\alpha}{d}.\frac{\beta|\gamma|\delta}{a|b|c}$$

$$\frac{\alpha|\beta|\gamma|\delta|\epsilon}{a|b|c|d|e} = \overset{\alpha}{a}.\frac{\beta|\gamma|\delta|\epsilon}{b|c|d|e} + \overset{\alpha}{b}.\frac{\beta|\gamma|\delta|\epsilon}{c|d|e|a} + \overset{\alpha}{c}.\frac{\beta|\gamma|\delta|\epsilon}{d|e|a|b} + \overset{\alpha}{d}.\frac{\beta|\gamma|\delta|\epsilon}{e|a|b|c} + \overset{\alpha}{e}.\frac{\beta|\gamma|\delta|\epsilon}{a|b|c|d}$$

$$\frac{\alpha|\beta|\gamma|\delta|\epsilon|\zeta}{a|b|c|d|e|f} = \overset{\alpha}{a}.\frac{\beta|\gamma|\delta|\epsilon|\zeta}{b|c|d|e|f} - \overset{\alpha}{b}.\frac{\beta|\gamma|\delta|\epsilon|\zeta}{c|d|e|f|a} + \text{&c.}$$

$\overset{\alpha}{a}$ denotes a single quantity, e.g., a coefficient in a linear equation

Define: $\dfrac{\alpha \mid \beta}{a \mid b} = \overset{\alpha}{a}\overset{\beta}{b} - \overset{\alpha}{b}\overset{\beta}{a}$

Anachronistically, this is the determinant of the matrix:

$$\begin{pmatrix} \alpha & \alpha \\ a & b \\ \\ \beta & \beta \\ a & b \end{pmatrix}$$

Then continue recursively . . .

Determinants

Leibniz, unpublished works, 1680s/1690s.

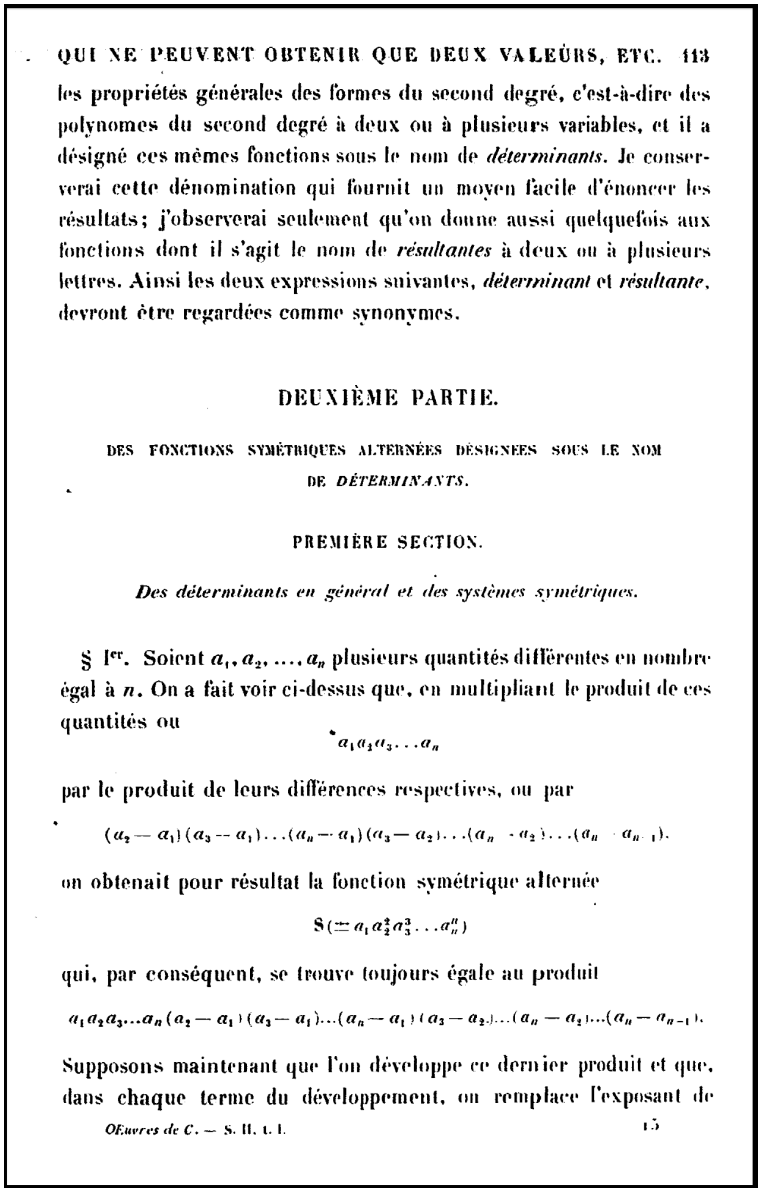
Seki Takakazu, *Kai-fukudai-no-hō* 解伏題之法, 1683

Colin Maclaurin, *A treatise of algebra*, 1748, Ch. XII, pp. 81–85

Vandermonde, 'Mémoire sur l'élimination', *Mémoires de l'Académie des sciences*, 1772: a recursive description of determinants of any size (but without a name and in an uncongenial notation — see *Mathematics emerging*, §17.1.3)

Gauss in *Disquisitiones arithmeticae* (1801) gave the name 'determinant' to what is now called the 'discriminant' $B^2 - AC$ of the binary quadratic form $Ax^2 + 2Bxy + Cy^2$.

Cauchy on determinants



QUI NE PEUVENT OBTENIR QUE DEUX VALEURS, ETC. 113

les propriétés générales des formes du second degré, c'est-à-dire des polynomes du second degré à deux ou à plusieurs variables, et il a désigné ces mêmes fonctions sous le nom de *déterminants*. Je conserverai cette dénomination qui fournit un moyen facile d'énoncer les résultats; j'observerai seulement qu'on donne aussi quelquefois aux fonctions dont il s'agit le nom de *résultantes* à deux ou à plusieurs lettres. Ainsi les deux expressions suivantes, *déterminant* et *résultante*, devront être regardées comme synonymes.

DEUXIÈME PARTIE.

DES FONCTIONS SYMÉTRIQUES ALTERNÉES DÉSIGNÉES SOUS LE NOM DE *DÉTERMINANTS*.

PREMIÈRE SECTION.

Des déterminants en général et des systèmes symétriques.

§ I^er^. Soient $a_1, a_2, \ldots, a_n$ plusieurs quantités différentes en nombre égal à n. On a fait voir ci-dessus que, en multipliant le produit de ces quantités ou

$$a_1 a_2 a_3 \ldots a_n$$

par le produit de leurs différences respectives, ou par

$$(a_2 - a_1)(a_3 - a_1)\ldots(a_n - a_1)(a_3 - a_2)\ldots(a_n - a_2)\ldots(a_n - a_{n-1}),$$

on obtenait pour résultat la fonction symétrique alternée

$$\mathrm{S}(\pm a_1 a_2^2 a_3^3 \ldots a_n^n)$$

qui, par conséquent, se trouve toujours égale au produit

$$a_1 a_2 a_3 \ldots a_n (a_2 - a_1)(a_3 - a_1)\ldots(a_n - a_1)(a_3 - a_2)\ldots(a_n - a_2)\ldots(a_n - a_{n-1}).$$

Supposons maintenant que l'on développe ce dernier produit et que, dans chaque terme du développement, on remplace l'exposant de

Œuvres de C. — S. II, t. I. 15

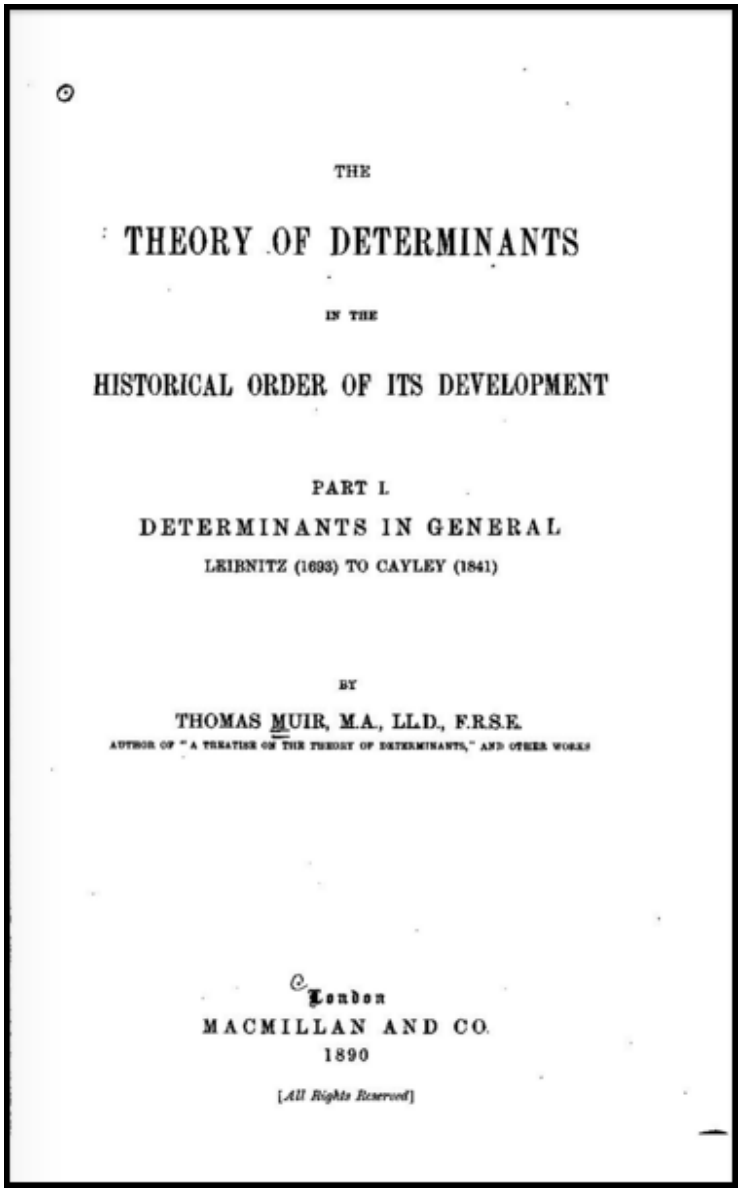
Cauchy, 'Mémoire sur les fonctions qui ne peuvent obtenir que deux valeurs égales et de signes contraires par suite des transpositions opérées entre les variables qu'elles renferment', *Journal de l'École polytechnique*, 1815

Referred to Laplace, Vandermonde, Gauss, and others

Introduced the term determinant for the function of n^2 quantities (a sum of $n!$ signed products) that we now know by that name.

(See *Mathematics emerging*, §17.1.4.)

History of the theory of determinants



THE

THEORY OF DETERMINANTS

IN THE

HISTORICAL ORDER OF ITS DEVELOPMENT

PART I.

DETERMINANTS IN GENERAL

LEIBNITZ (1693) TO CAYLEY (1841)

BY

THOMAS MUIR, M.A., LL.D., F.R.S.E.

AUTHOR OF "A TREATISE ON THE THEORY OF DETERMINANTS," AND OTHER WORKS

London

MACMILLAN AND CO.

1890

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Determinants were studied extensively in the 19th century.

Sir Thomas Muir, *The theory of determinants in the historical order of development* (1890–1906)

- Part I: *Determinants in general: Leibnitz (1693) to Cayley (1841)*;
- Part II: *Special determinants up to 1841*

Second edition in 4 volumes, 1906–1923; supplement, 1930.

'Eigenvalue' problems

Euler (1748): change of coordinates to reduce equation of a quadric surface $\alpha z^2 + \beta yz + \gamma xz + \delta y^2 + \epsilon xy + \zeta x^2 + \eta z + \theta y + \iota x + \chi = 0$ to its simplest form $Ap^2 + Bq^2 + Cr^2 + K = 0$ (see: *Mathematics emerging*, §17.2.1.)

Laplace (1787): symmetry of coefficients in a set of linear differential equations leads to real 'eigenvalues' (see: *Mathematics emerging*, §17.2.2.)

Cauchy (1829): a symmetric matrix is diagonalisable by a real orthogonal change of variables (see: *Mathematics emerging*, §17.2.3.)

Matrices and their determinants

Gauss, *Disquisitiones arithmeticae* (1801): transformation of quadratic forms $ax^2 + 2bxy + cy^2$ by change of variables

$$x = \alpha x' + \beta y', \quad y = \gamma x' + \delta y'$$

followed by

$$x' = \alpha' x'' + \beta' y'', \quad y' = \gamma' x'' + \delta' y''$$

comes to the same as

$$x = (\alpha\alpha' + \beta\gamma')x'' + (\alpha\beta' + \beta\delta')y'', \quad y = (\gamma\alpha' + \delta\gamma')x'' + (\gamma\beta' + \delta\delta')y''$$

Moreover, the 'determinants' (our sense) multiply.

NB. All Gauss' coefficients were integers

(See *Mathematics emerging*, §17.3.1.)

Early origins of matrices

The OED (3rd ed., March 2001) lists sense 2a of 'matrix' as

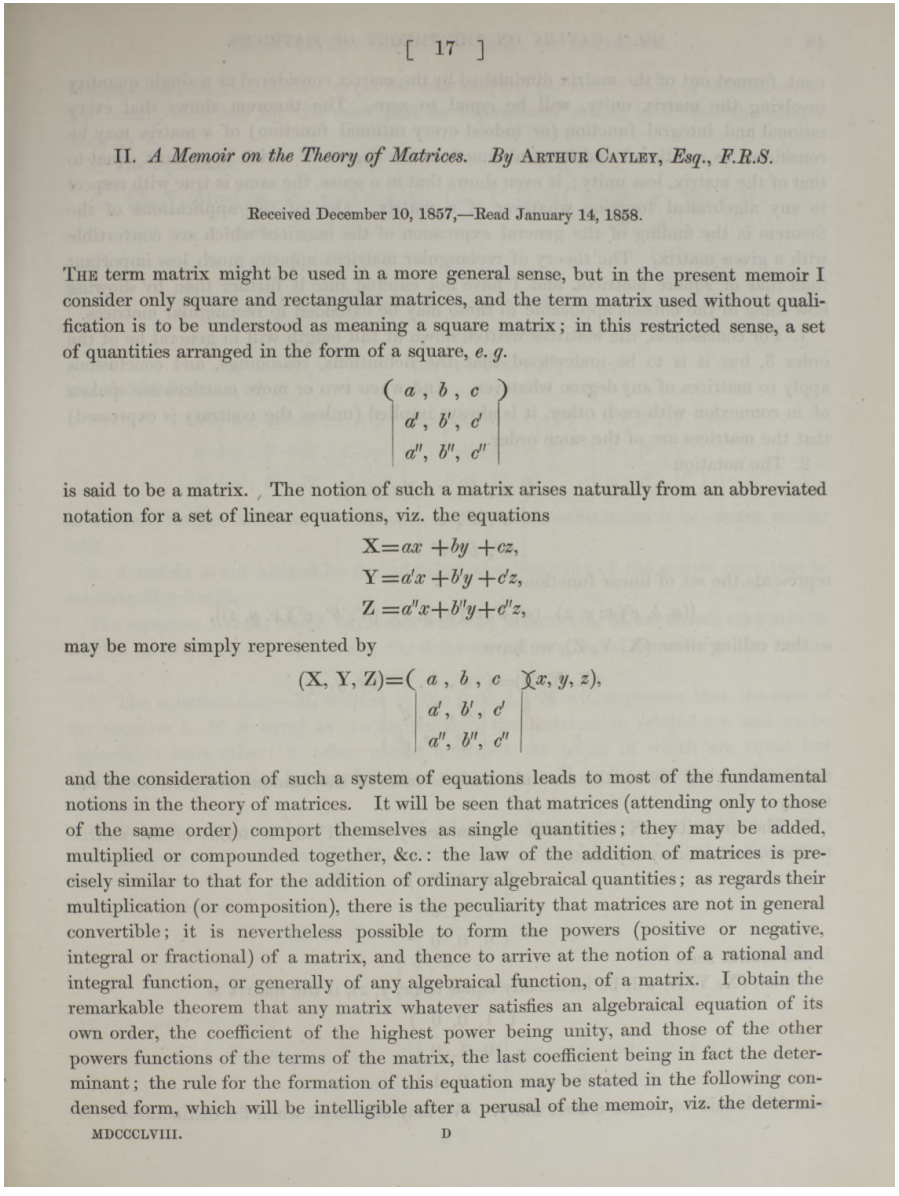
> *A place or medium in which something is originated, produced, or developed ...*

Thus, in 1850, J. J. Sylvester applied the word to the 'thing' from which determinants originate:

> *For this purpose we must commence, not with a square, but with an oblong arrangement of terms consisting, suppose, of m lines and n columns. This will not in itself represent a determinant, but is, as it were, a Matrix out of which we may form various systems of determinants by fixing upon a number p, and selecting at will p lines and p columns, the squares corresponding of pth order.*

But he did not operate with matrices

The definition of matrices



[17]

II. *A Memoir on the Theory of Matrices.* By Arthur Cayley, *Esq.*, *F.R.S.*

Received December 10, 1857,—Read January 14, 1858.

The term matrix might be used in a more general sense, but in the present memoir I consider only square and rectangular matrices, and the term matrix used without qualification is to be understood as meaning a square matrix; in this restricted sense, a set of quantities arranged in the form of a square, *e. g.*

$$\begin{pmatrix} a, & b, & c \\ a', & b', & c' \\ a'', & b'', & c'' \end{pmatrix}$$

is said to be a matrix. The notion of such a matrix arises naturally from an abbreviated notation for a set of linear equations, viz. the equations

$$\begin{aligned} X &= ax + by + cz, \\ Y &= a'x + b'y + c'z, \\ Z &= a''x + b''y + c''z, \end{aligned}$$

may be more simply represented by

$$(X, Y, Z) = \begin{pmatrix} a, & b, & c \\ a', & b', & c' \\ a'', & b'', & c'' \end{pmatrix} (x, y, z),$$

and the consideration of such a system of equations leads to most of the fundamental notions in the theory of matrices. It will be seen that matrices (attending only to those of the same order) comport themselves as single quantities; they may be added, multiplied or compounded together, &c.: the law of the addition of matrices is precisely similar to that for the addition of ordinary algebraical quantities; as regards their multiplication (or composition), there is the peculiarity that matrices are not in general convertible; it is nevertheless possible to form the powers (positive or negative, integral or fractional) of a matrix, and thence to arrive at the notion of a rational and integral function, or generally of any algebraical function, of a matrix. I obtain the remarkable theorem that any matrix whatever satisfies an algebraical equation of its own order, the coefficient of the highest power being unity, and those of the other powers functions of the terms of the matrix, the last coefficient being in fact the determinant; the rule for the formation of this equation may be stated in the following condensed form, which will be intelligible after a perusal of the memoir, viz. the determi-

MDCCCLVIII. D

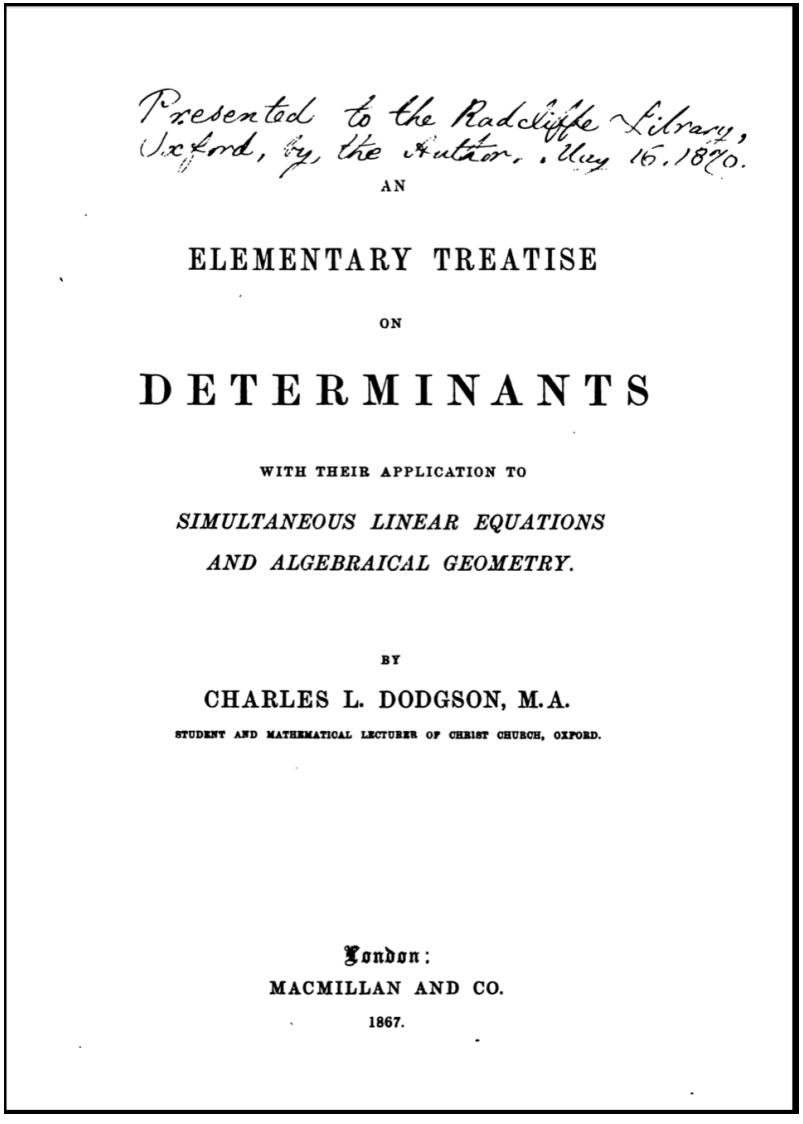
Arthur Cayley, 'A memoir on the theory of matrices', *Phil. Trans. Roy. Soc.*, 1858:

- defined matrices and their properties
- recognised connection to linear equations
- stated the Cayley–Hamilton Theorem
- investigated the matrices that commute with a given one

"It will be seen that matrices (attending only to those of the same order) comport themselves as single quantities..."

(See *Mathematics emerging*, §17.3.2.)

Determinants persist



Presented to the Radcliffe Library, Oxford, by the Author. May 16. 1870.

AN

ELEMENTARY TREATISE

ON

DETERMINANTS

WITH THEIR APPLICATION TO

SIMULTANEOUS LINEAR EQUATIONS AND ALGEBRAICAL GEOMETRY.

BY

CHARLES L. DODGSON, M.A.

STUDENT AND MATHEMATICAL LECTURER OF CHRIST CHURCH, OXFORD.

London:
MACMILLAN AND CO.
1867.

"I am aware that the word 'Matrix' is already in use to express the very meaning for which I use the word 'Block'; but surely the former word means rather the mould, or form, into which algebraical quantities may be introduced, than an actual assemblage of such quantities . . ."

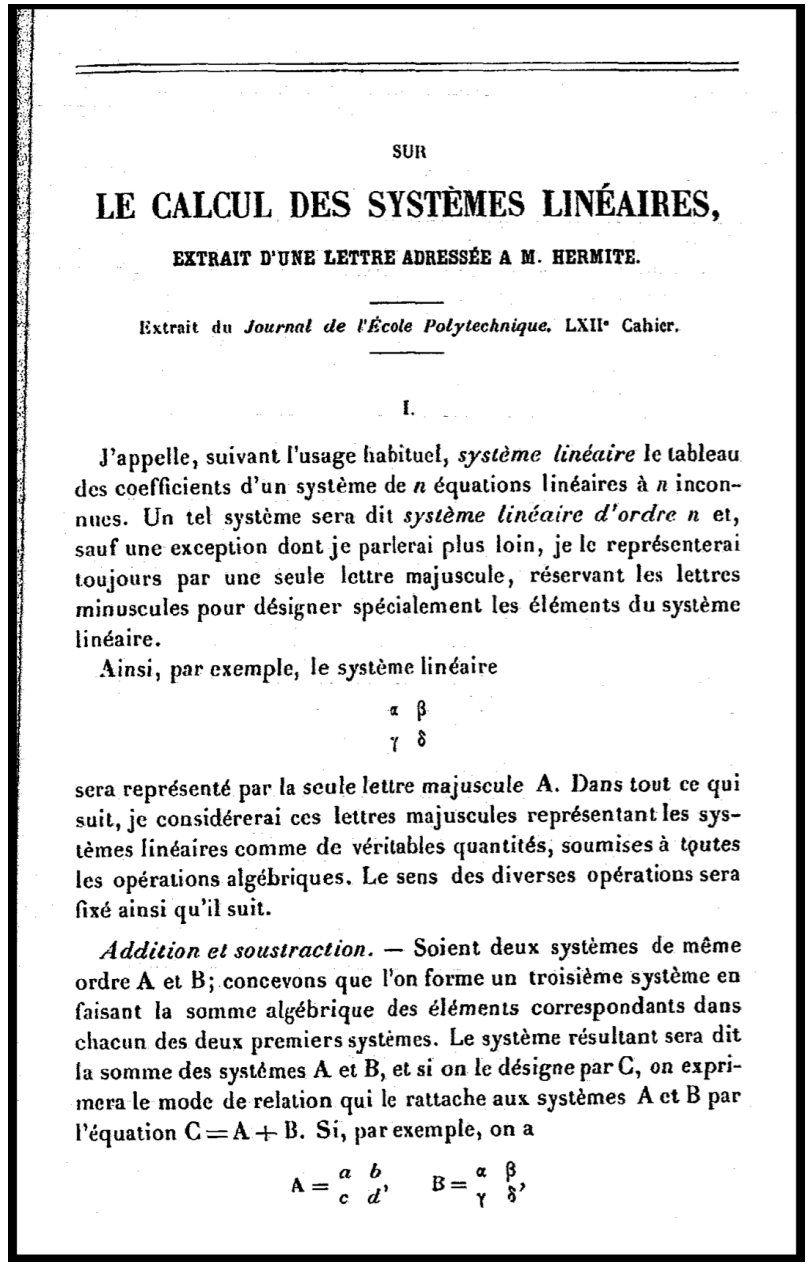
Criticised notation 'a_{ij}':
"it seems a fatal objection to this system that most of the space is occupied by a number of a's, which are wholly superfluous, while the only important part of the notation is reduced to minute subscripts, alike difficult to the writer and the reader."

Proposed $i \backslash j$ instead

Matrices elsewhere

Matrix algebra appears in Hamilton's *Lectures on Quaternions* (1853) as 'linear and vector functions' (including his version of the Cayley–Hamilton Theorem, stated and proved in terms of quaternions)

Matrices were also devised by Laguerre in his paper 'Sur le calcul des systèmes linéaires' (*J. École polytechnique*, 1867)



SUR

LE CALCUL DES SYSTÈMES LINÉAIRES,

EXTRAIT D'UNE LETTRE ADRESSÉE A M. HERMITE.

Extrait du *Journal de l'École Polytechnique*, LXII^e Cahier.

I.

J'appelle, suivant l'usage habituel, *système linéaire* le tableau des coefficients d'un système de n équations linéaires à n inconnues. Un tel système sera dit *système linéaire d'ordre n* et, sauf une exception dont je parlerai plus loin, je le représenterai toujours par une seule lettre majuscule, réservant les lettres minuscules pour désigner spécialement les éléments du système linéaire.

Ainsi, par exemple, le système linéaire

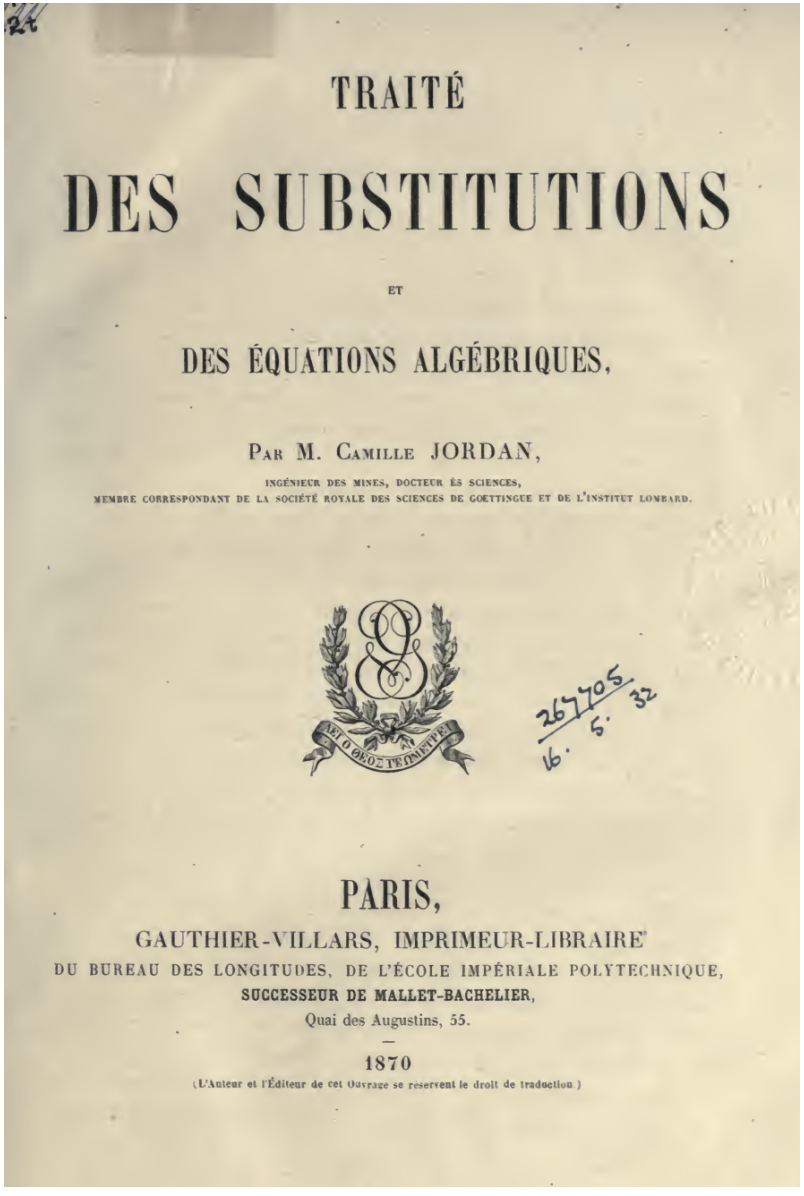
$$\begin{matrix} \alpha & \beta \\ \gamma & \delta \end{matrix}$$

sera représenté par la seule lettre majuscule A. Dans tout ce qui suit, je considérerai ces lettres majuscules représentant les systèmes linéaires comme de véritables quantités, soumises à toutes les opérations algébriques. Le sens des diverses opérations sera fixé ainsi qu'il suit.

Addition et soustraction. — Soient deux systèmes de même ordre A et B; concevons que l'on forme un troisième système en faisant la somme algébrique des éléments correspondants dans chacun des deux premiers systèmes. Le système résultant sera dit la somme des systèmes A et B, et si on le désigne par C, on exprimera le mode de relation qui le rattache aux systèmes A et B par l'équation $C = A + B$. Si, par exemple, on a

$$A = \begin{matrix} a & b \\ c & d \end{matrix}, \qquad B = \begin{matrix} \alpha & \beta \\ \gamma & \delta \end{matrix},$$

Jordan and linear substitutions



TRAITÉ

DES SUBSTITUTIONS

ET

DES ÉQUATIONS ALGÉBRIQUES,

PAR M. CAMILLE JORDAN,

INGÉNIEUR DES MINES, DOCTEUR ÈS SCIENCES,
MEMBRE CORRESPONDANT DE LA SOCIÉTÉ ROYALE DES SCIENCES DE GOETTINGUE ET DE L'INSTITUT LOMBARD.

267705
5. 32
16.

PARIS,

GAUTHIER-VILLARS, IMPRIMEUR-LIBRAIRE
DU BUREAU DES LONGITUDES, DE L'ÉCOLE IMPÉRIALE POLYTECHNIQUE,
SUCCESSEUR DE MALLET-BACHELIER,
Quai des Augustins, 55.

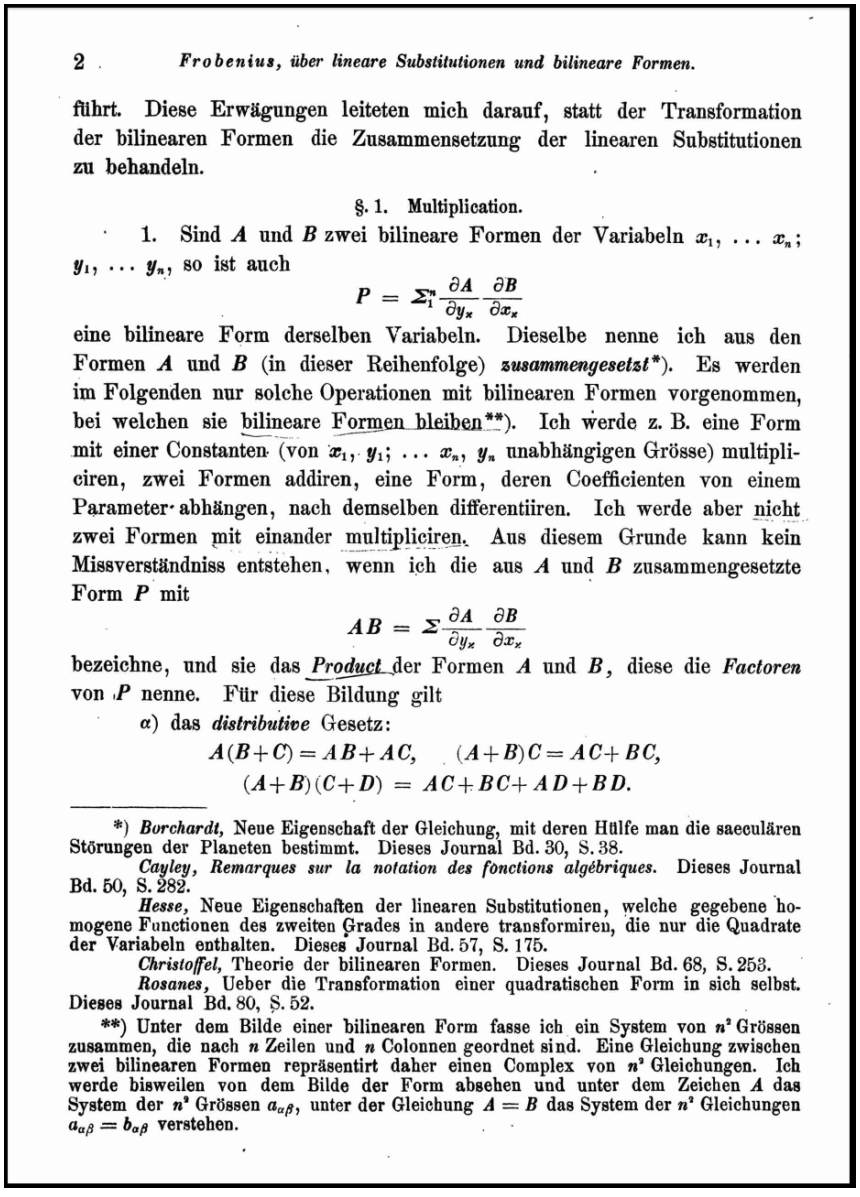
1870

(L'Auteur et l'Éditeur de cet Ouvrage se réservent le droit de traduction.)

Camille Jordan, *Traité des substitutions*, 1870:

- studied matrices over integers modulo n as part of an extensive study of linear substitutions (in connection with Galois theory); developed 'canonical forms' to study conjugacy classes in these groups
- developed his ideas to 'Jordan canonical form' for complex matrices in his studies 1872–4 of linear differential equations

German contributions



2 *Frobenius, über lineare Substitutionen und bilineare Formen.*

führt. Diese Erwägungen leiteten mich darauf, statt der Transformation der bilinearen Formen die Zusammensetzung der linearen Substitutionen zu behandeln.

§. 1. Multiplication.

1. Sind A und B zwei bilineare Formen der Variabeln $x_1, \dots x_n$; $y_1, \dots y_n$, so ist auch

$$P = \Sigma_1^n \frac{\partial A}{\partial y_\varkappa} \frac{\partial B}{\partial x_\varkappa}$$

eine bilineare Form derselben Variabeln. Dieselbe nenne ich aus den Formen A und B (in dieser Reihenfolge) ***zusammengesetzt****). Es werden im Folgenden nur solche Operationen mit bilinearen Formen vorgenommen, bei welchen sie bilineare Formen bleiben*****). Ich werde z. B. eine Form mit einer Constanten (von $x_1, y_1; \dots x_n, y_n$ unabhängigen Grösse) multipliciren, zwei Formen addiren, eine Form, deren Coefficienten von einem Parameter abhängen, nach demselben differentiiren. Ich werde aber nicht zwei Formen mit einander multipliciren. Aus diesem Grunde kann kein Missverständniss entstehen, wenn ich die aus A und B zusammengesetzte Form P mit

$$AB = \Sigma \frac{\partial A}{\partial y_\varkappa} \frac{\partial B}{\partial x_\varkappa}$$

bezeichne, und sie das *Product* der Formen A und B, diese die *Factoren* von P nenne. Für diese Bildung gilt

α) das *distributive* Gesetz:

$$A(B+C) = AB+AC, \quad (A+B)C = AC+BC,$$
$$(A+B)(C+D) = AC+BC+AD+BD.$$

*) *Borchardt*, Neue Eigenschaft der Gleichung, mit deren Hülfe man die saeculären Störungen der Planeten bestimmt. Dieses Journal Bd. 30, S. 38.
Cayley, Remarques sur la notation des fonctions algébriques. Dieses Journal Bd. 50, S. 282.
Hesse, Neue Eigenschaften der linearen Substitutionen, welche gegebene homogene Functionen des zweiten Grades in andere transformiren, die nur die Quadrate der Variabeln enthalten. Dieses Journal Bd. 57, S. 175.
Christoffel, Theorie der bilinearen Formen. Dieses Journal Bd. 68, S. 253.
Rosanes, Ueber die Transformation einer quadratischen Form in sich selbst. Dieses Journal Bd. 80, S. 52.

**) Unter dem Bilde einer bilinearen Form fasse ich ein System von n^2 Grössen zusammen, die nach n Zeilen und n Colonnen geordnet sind. Eine Gleichung zwischen zwei bilinearen Formen repräsentirt daher einen Complex von n^2 Gleichungen. Ich werde bisweilen von dem Bilde der Form absehen und unter dem Zeichen A das System der n^2 Grössen $a_{\alpha\beta}$, unter der Gleichung $A = B$ das System der n^2 Gleichungen $a_{\alpha\beta} = b_{\alpha\beta}$ verstehen.

Georg Frobenius, in 1878, working with bilinear forms, produced more canonical forms, and gave a satisfactory proof of the Cayley–Hamilton Theorem

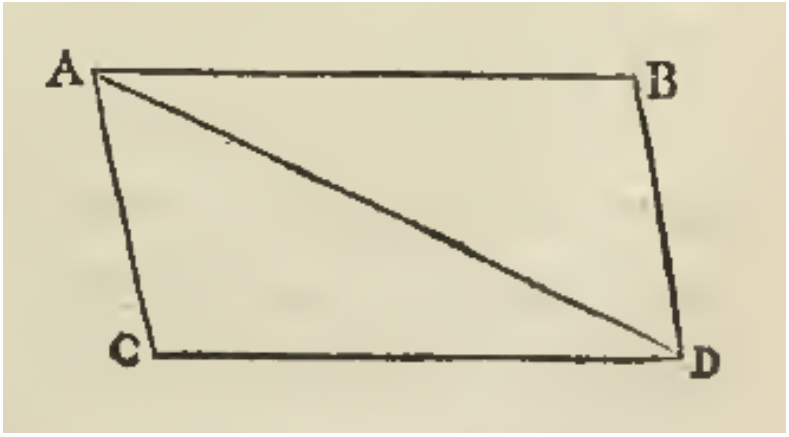
(See *Mathematics emerging*, §17.3.3.)

Other mathematicians in Germany (e.g., Kronecker, Hurwitz) contributed similarly

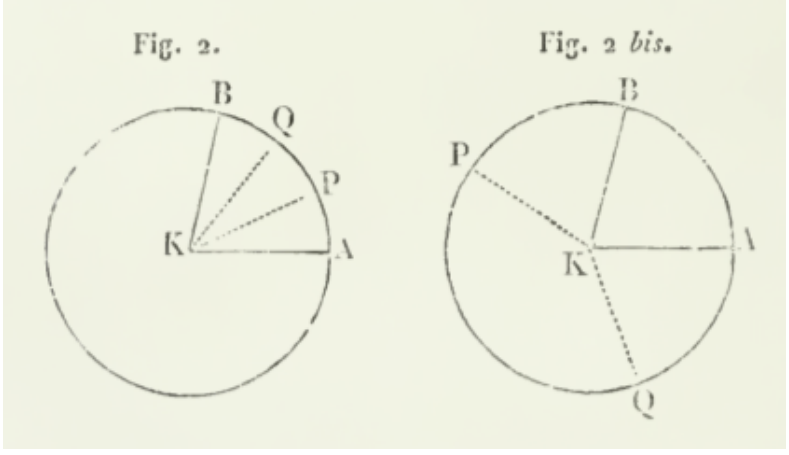
A recommended secondary source: Thomas Hawkins, 'Another look at Cayley and the theory of matrices', *Archives internationales d'histoire des sciences* **26** (1977), 82–112

Vectors

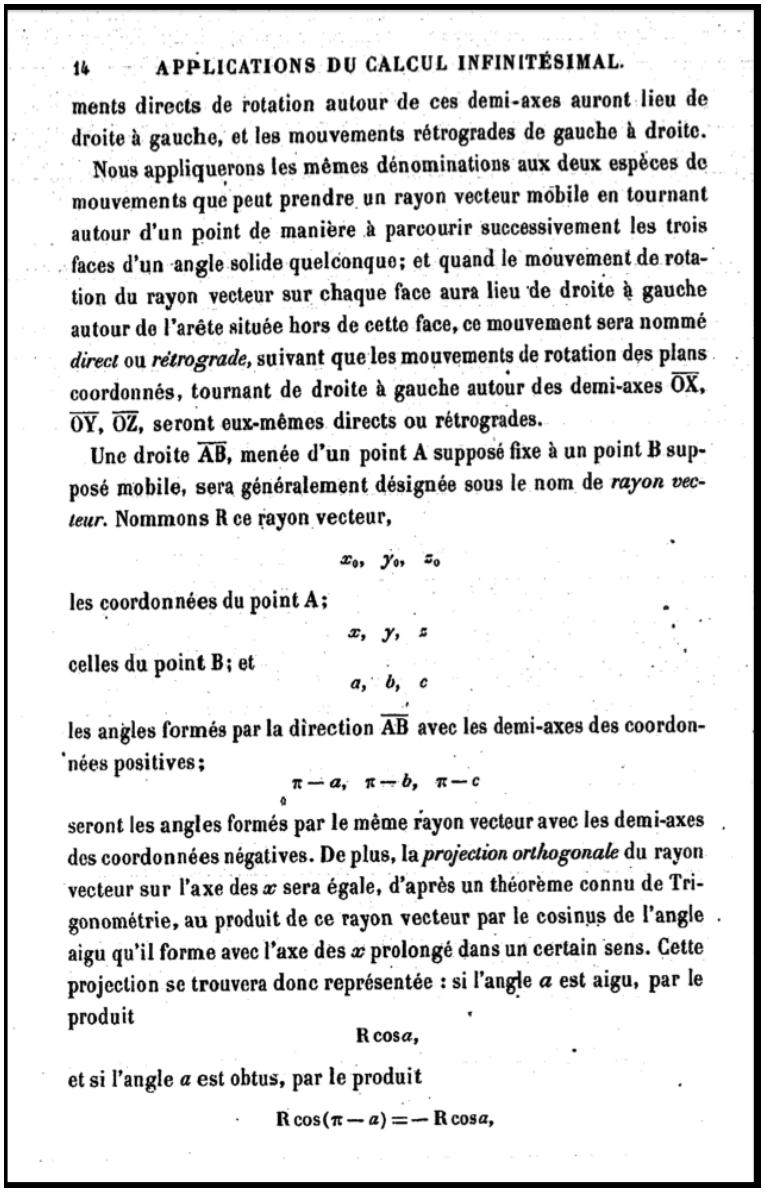
Newton (1687): parallelogram of forces



Argand (1806): complex numbers as directed quantities in the plane



Vectors



14 APPLICATIONS DU CALCUL INFINITÉSIMAL.

ments directs de rotation autour de ces demi-axes auront lieu de droite à gauche, et les mouvements rétrogrades de gauche à droite.

Nous appliquerons les mêmes dénominations aux deux espèces de mouvements que peut prendre un rayon vecteur mobile en tournant autour d'un point de manière à parcourir successivement les trois faces d'un angle solide quelconque; et quand le mouvement de rotation du rayon vecteur sur chaque face aura lieu de droite à gauche autour de l'arête située hors de cette face, ce mouvement sera nommé *direct* ou *rétrograde*, suivant que les mouvements de rotation des plans coordonnés, tournant de droite à gauche autour des demi-axes $\overline{OX}$, $\overline{OY}$, $\overline{OZ}$, seront eux-mêmes directs ou rétrogrades.

Une droite $\overline{AB}$, menée d'un point A supposé fixe à un point B supposé mobile, sera généralement désignée sous le nom de *rayon vecteur*. Nommons R ce rayon vecteur,

$$x_0, \quad y_0, \quad z_0$$

les coordonnées du point A;

$$x, \quad y, \quad z$$

celles du point B; et

$$a, \quad b, \quad c$$

les angles formés par la direction $\overline{AB}$ avec les demi-axes des coordonnées positives;

$$\pi - a, \quad \pi - b, \quad \pi - c$$

seront les angles formés par le même rayon vecteur avec les demi-axes des coordonnées négatives. De plus, la *projection orthogonale* du rayon vecteur sur l'axe des x sera égale, d'après un théorème connu de Trigonométrie, au produit de ce rayon vecteur par le cosinus de l'angle aigu qu'il forme avec l'axe des x prolongé dans un certain sens. Cette projection se trouvera donc représentée : si l'angle a est aigu, par le produit

$$\mathrm{R}\cos a,$$

et si l'angle a est obtus, par le produit

$$\mathrm{R}\cos(\pi - a) = -\mathrm{R}\cos a,$$

Word applied mostly to radius vectors

e.g., as rayon vecteur in Laplace's *Mécanique Céleste* (1799–1825)

Also in Cauchy's *Leçons sur les Applications du Calcul Infinitésimal à la Géométrie* (1826), p. 14:

A line $\overline{\mathrm{AB}}$, *taken from a point* A, *supposed to be fixed, to a moving point* B, *will in general be referred to as a radius vector.*

Hamilton and vectors

Sir William Rowan Hamilton drew a distinction between a 'vector' and a 'radius vector':

Between 1843–1866, developed quaternions — 4-dimensional quantities $a + bi + cj + dk$, where $i^2 = j^2 = k^2 = ijk = -1$, designed for use in mechanics (and geometry of 3 dimensions)

"*A* VECTOR *is thus . . . a sort of* NATURAL TRIPLET *(suggested by Geometry): and accordingly we shall find that* QUATERNIONS *offer an easy mode of symbolically representing every vector by a* TRINOMIAL FORM *(*$ix + jy + kz$*); which form brings the conception and expression of such a vector into the closest possible connexions with Cartesian and rectangular co-ordinates.*"

So a quaternion is a scalar + a vector

Vector spaces appear

Die Alexander Ziwet

Ausdehnungslehre.

Vollständig und in strenger Form

bearbeitet

von

Hermann Grassmann,

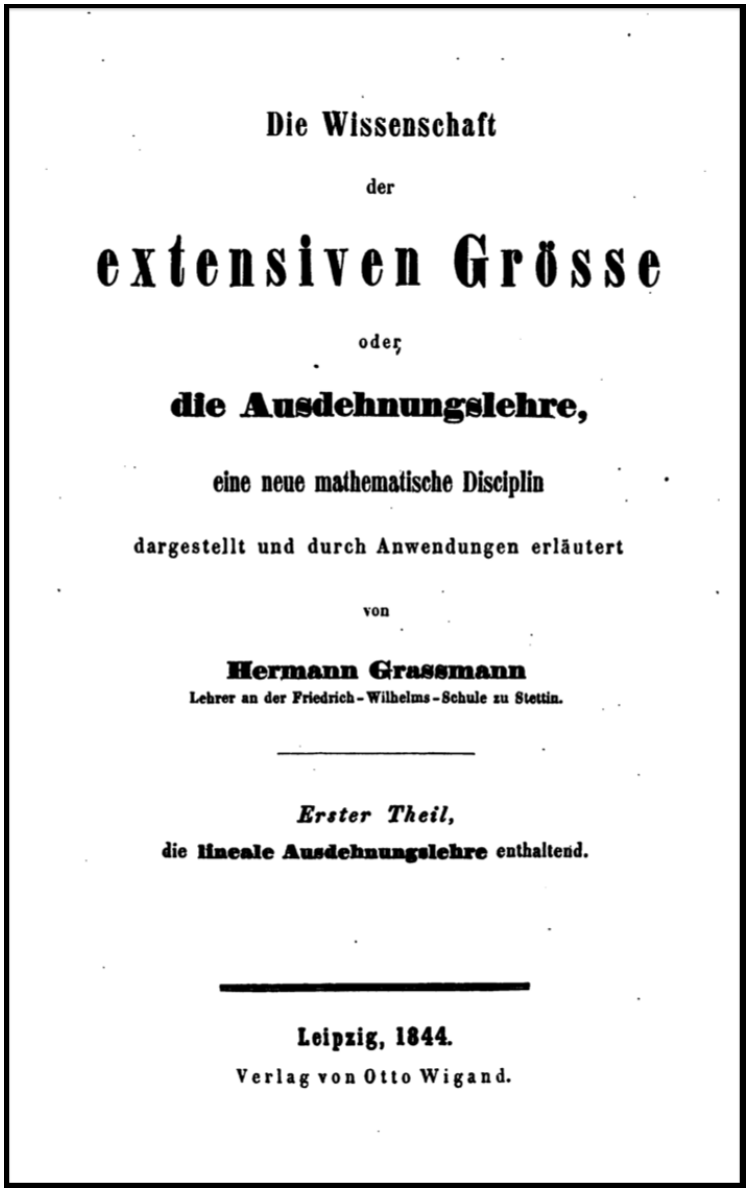
Professor am Gymnasium zu Stettin.

BERLIN, 1862.

VERLAG VON TH. CHR. FR. ENSLIN.

(ADOLPH ENSLIN.)

Grassmann's 'doctrine of extension'



Die Wissenschaft

der

extensiven Grösse

oder

die Ausdehnungslehre,

eine neue mathematische Disciplin

dargestellt und durch Anwendungen erläutert

von

Hermann Grassmann

Lehrer an der Friedrich-Wilhelms-Schule zu Stettin.

Erster Theil,

die lineale Ausdehnungslehre enthaltend.

Leipzig, 1844.

Verlag von Otto Wigand.

Die Ausdehnungslehre [*Doctrine of extension*] (1862) is a heavily reworked version of an earlier (1844) work:

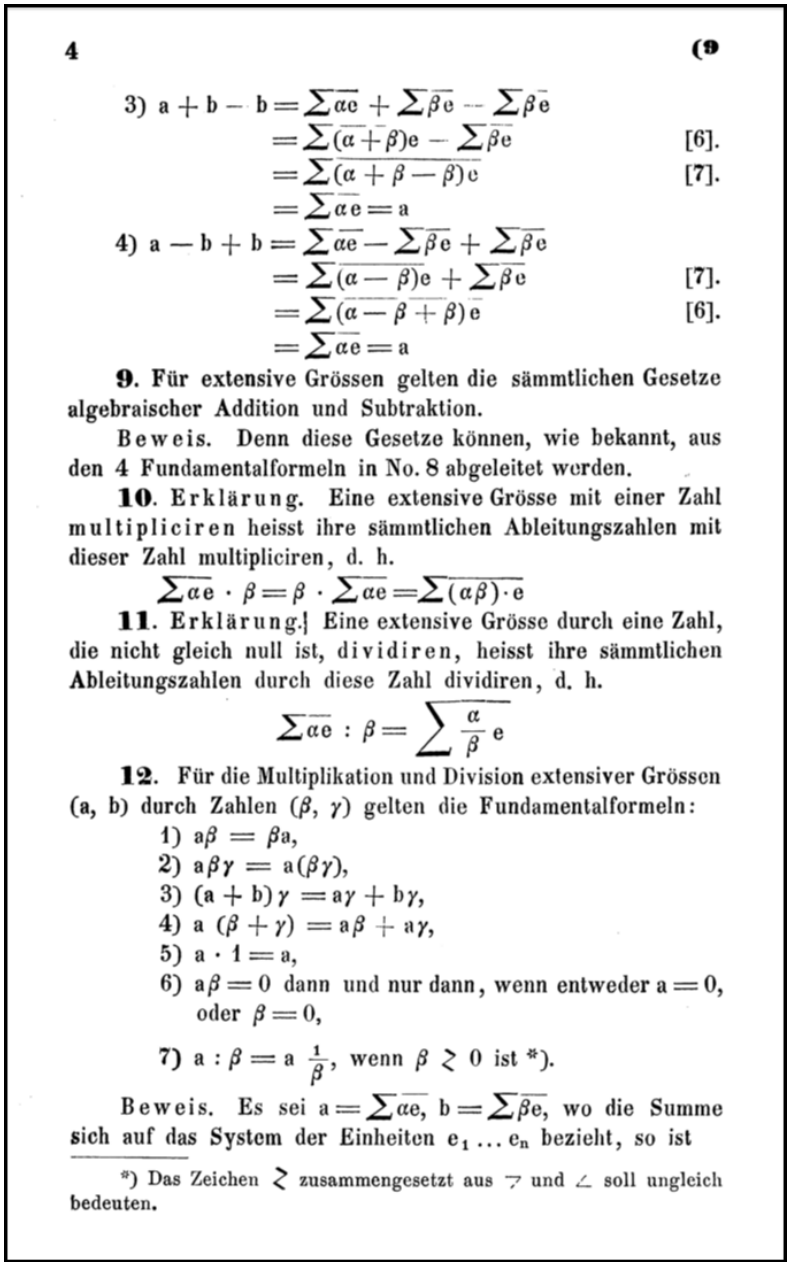
The Science of Extensive Quantities, or the Doctrine of Extension, a New Mathematical Discipline, Presented and Explained through Examples

Introduced idea of objects generated by motion — a single element generates an object of order 1, an object of order 1 generates an object of order 2, etc.

Objects of the same order can be added together or scaled by real numbers

Little impact at the time

Grassmann's 'extensive quantities'



4 (9

3) $a + b - b = \sum \overline{\alpha e} + \sum \overline{\beta e} - \sum \overline{\beta e}$
$= \sum \overline{(\alpha + \beta) e} - \sum \overline{\beta e}$ [6].
$= \sum \overline{(\alpha + \beta - \beta) e}$ [7].
$= \sum \overline{\alpha e} = a$

4) $a - b + b = \sum \overline{\alpha e} - \sum \overline{\beta e} + \sum \overline{\beta e}$
$= \sum \overline{(\alpha - \beta) e} + \sum \overline{\beta e}$ [7].
$= \sum \overline{(\alpha - \beta + \beta) e}$ [6].
$= \sum \overline{\alpha e} = a$

9. Für extensive Grössen gelten die sämmtlichen Gesetze algebraischer Addition und Subtraktion.

Beweis. Denn diese Gesetze können, wie bekannt, aus den 4 Fundamentalformeln in No. 8 abgeleitet werden.

10. Erklärung. Eine extensive Grösse mit einer Zahl multipliciren heisst ihre sämmtlichen Ableitungszahlen mit dieser Zahl multipliciren, d. h.

$$\sum \overline{\alpha e} \cdot \beta = \beta \cdot \sum \overline{\alpha e} = \sum \overline{(\alpha\beta) \cdot e}$$

11. Erklärung. Eine extensive Grösse durch eine Zahl, die nicht gleich null ist, dividiren, heisst ihre sämmtlichen Ableitungszahlen durch diese Zahl dividiren, d. h.

$$\sum \overline{\alpha e} : \beta = \sum \overline{\frac{\alpha}{\beta} e}$$

12. Für die Multiplikation und Division extensiver Grössen (a, b) durch Zahlen (β, γ) gelten die Fundamentalformeln:

1) $a\beta = \beta a$,
2) $a\beta\gamma = a(\beta\gamma)$,
3) $(a + b)\gamma = a\gamma + b\gamma$,
4) $a(\beta + \gamma) = a\beta + a\gamma$,
5) $a \cdot 1 = a$,
6) $a\beta = 0$ dann und nur dann, wenn entweder $a = 0$, oder $\beta = 0$,
7) $a : \beta = a \frac{1}{\beta}$, wenn $\beta \gtrless 0$ ist *).

Beweis. Es sei $a = \sum \overline{\alpha e}$, $b = \sum \overline{\beta e}$, wo die Summe sich auf das System der Einheiten $e_1 \ldots e_n$ bezieht, so ist

*) Das Zeichen $\gtrless$ zusammengesetzt aus $\gtrless$ und $\lessgtr$ soll ungleich bedeuten.

The 1862 text contains a theory of extensive quantities

$$a_1 e_1 + a_2 e_2 + \cdots ,$$

where the e_i are 'units' and the a_i are real numbers, including

- rules for the arithmetic of such quantities
- a notion of linear independence
- dimension
- ...

But still had little impact

(See *Mathematics emerging*, §17.4.1.)

Vector spaces defined

CALCOLO

GEOMETRICO

secondo l'Ausdehnungslehre di H. Grassmann

PRECEDUTO DALLE

OPERAZIONI DELLA LOGICA DEDUTTIVA

PER

GIUSEPPE PEANO

PROFESSORE NELLA R. ACCADEMIA MILITARE
INCARICATO NELLA R. UNIVERSITÀ DI TORINO

TORINO
FRATELLI BOCCA EDITORI
LIBRAI DI S. M. IL RE D'ITALIA

SUCCURSALI
ROMA
Via del Corso, 216-217.
FIRENZE
Via Cerretani, 8

DEPOSITI
PALERMO
Università, 12
(N. Carosio)
NAPOLI
Piazza Plebiscito, 2
CATANIA
S. Maria al Ros.o, 23
(N. Carosio)

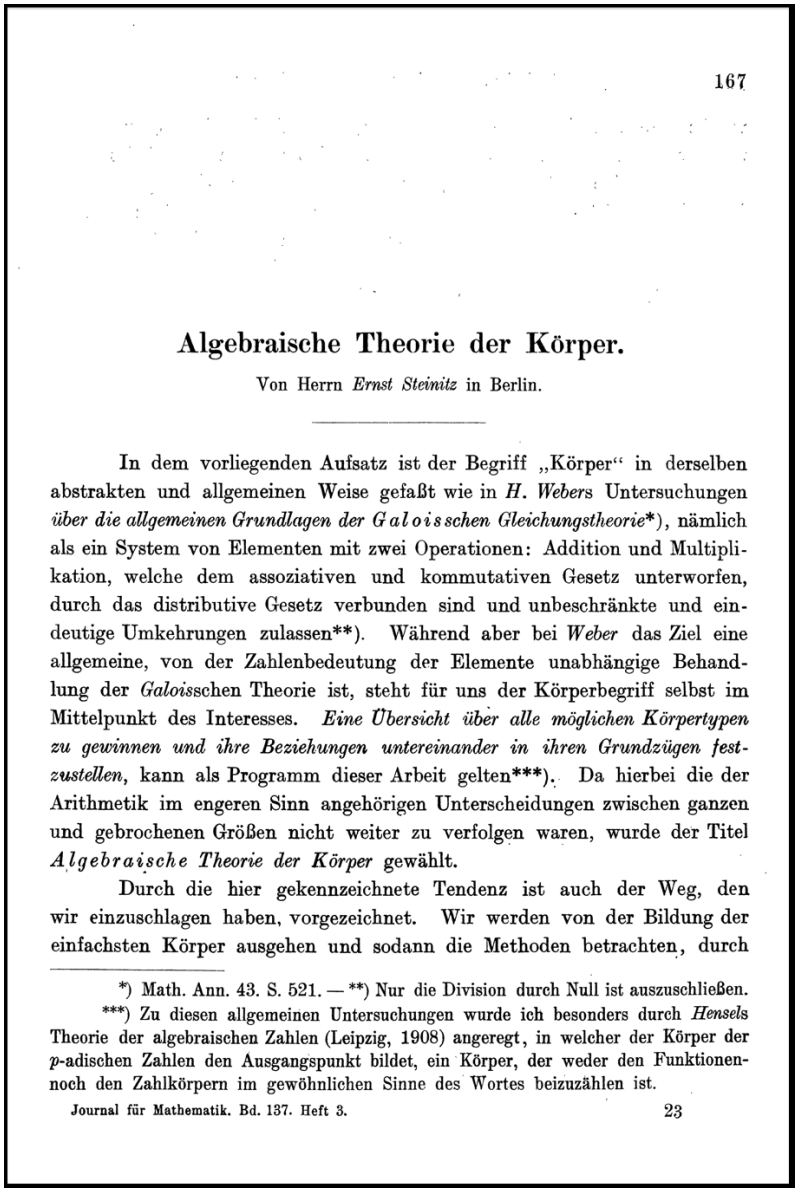
1888

On the way towards developing a 'geometric calculus', Guiseppe Peano axiomatised Grassmann's collections of extensive quantities as linear systems (sistemi lineari), and moved to a fully abstract setting

Clarified connection between dimension and linear independence — noted existence of linear systems with infinite dimension

Also no immediate impact!

Vector spaces develop



167

Algebraische Theorie der Körper.

Von Herrn *Ernst Steinitz* in Berlin.

In dem vorliegenden Aufsatz ist der Begriff „Körper“ in derselben abstrakten und allgemeinen Weise gefaßt wie in *H. Webers* Untersuchungen *über die allgemeinen Grundlagen der Galoisschen Gleichungstheorie**), nämlich als ein System von Elementen mit zwei Operationen: Addition und Multiplikation, welche dem assoziativen und kommutativen Gesetz unterworfen, durch das distributive Gesetz verbunden sind und unbeschränkte und eindeutige Umkehrungen zulassen**). Während aber bei *Weber* das Ziel eine allgemeine, von der Zahlenbedeutung der Elemente unabhängige Behandlung der *Galois*schen Theorie ist, steht für uns der Körperbegriff selbst im Mittelpunkt des Interesses. *Eine Übersicht über alle möglichen Körpertypen zu gewinnen und ihre Beziehungen untereinander in ihren Grundzügen festzustellen*, kann als Programm dieser Arbeit gelten***). Da hierbei die der Arithmetik im engeren Sinn angehörigen Unterscheidungen zwischen ganzen und gebrochenen Größen nicht weiter zu verfolgen waren, wurde der Titel *Algebraische Theorie der Körper* gewählt.

Durch die hier gekennzeichnete Tendenz ist auch der Weg, den wir einzuschlagen haben, vorgezeichnet. Wir werden von der Bildung der einfachsten Körper ausgehen und sodann die Methoden betrachten, durch

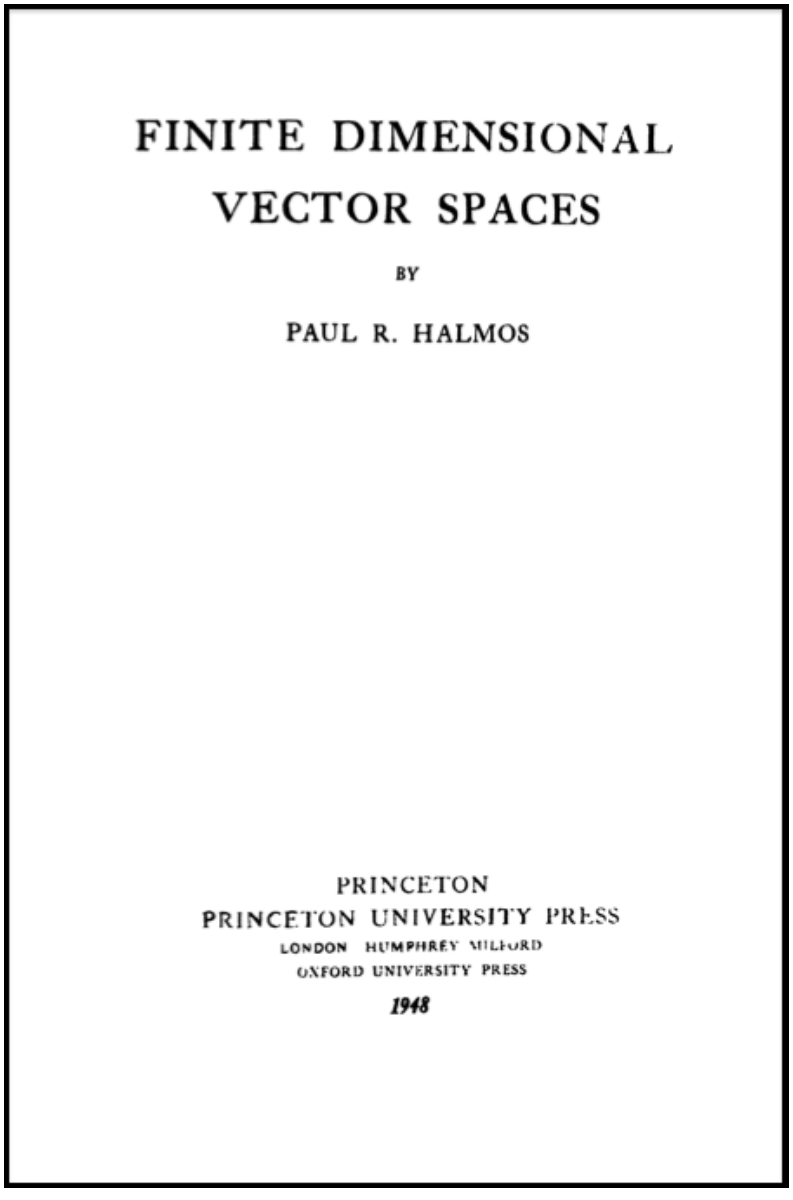
*) Math. Ann. 43. S. 521. — **) Nur die Division durch Null ist auszuschließen.
***) Zu diesen allgemeinen Untersuchungen wurde ich besonders durch *Hensels* Theorie der algebraischen Zahlen (Leipzig, 1908) angeregt, in welcher der Körper der p-adischen Zahlen den Ausgangspunkt bildet, ein Körper, der weder den Funktionen- noch den Zahlkörpern im gewöhnlichen Sinne des Wortes beizuzählen ist.

Journal für Mathematik. Bd. 137. Heft 3. 23

Dedekind (1879): fields and ‘modules’ needed for algebraic number theory in famous appendices to his third edition of Dirichlet, *Vorlesungen über Zahlentheorie* [*Lectures on number theory*]; published also separately in France, 1876–77

Ernst Steinitz (1910), ‘Algebraische Theorie der Körper’ [‘Algebraic theory of fields’] — contains a beautifully crystallised theory of linear dependence and independence, bases, dimension, etc., in the form it is now taught

Vector spaces develop


FINITE DIMENSIONAL
VECTOR SPACES

BY

PAUL R. HALMOS

PRINCETON
PRINCETON UNIVERSITY PRESS
LONDON HUMPHREY MILFORD
OXFORD UNIVERSITY PRESS
1948

B. L. van der Waerden (1930–31), *Moderne Algebra*, incorporating material from lectures by Emil Artin and Emmy Noether (1926–1928)

Paul Halmos (1942), *Finite-dimensional vector spaces* — made the subject accessible to 1st and 2nd year undergraduates

BO1 History of Mathematics
Lecture XV
Geometry and number theory

MT 2022 Week 8

Summary

- Euclid's *Elements* revisited
- The parallel postulate
- Non-Euclidean geometry
- Number theory down the centuries

Euclid's *Elements*

Euclid's Elements, in 13 books, compiled c. 250 BC.

Books I–V:	definitions, postulates, plane geometry of lines and circles
Book VI:	similarity, proportion
Books VII–IX:	number theory
Book X:	commensurability, irrational numbers, surds
Books XI–XIII:	solid geometry ending with the classification of the regular polyhedra

Euclid's *Elements*

Euclid's Elements, in 13 books, compiled c. 250 BC.

Books I–V:	definitions, postulates, plane geometry of lines and circles
Book VI:	similarity, proportion
Books VII–IX:	number theory
Book X:	commensurability, irrational numbers, surds
Books XI–XIII:	solid geometry ending with the classification of the regular polyhedra

Euclid in English

BOOK I.

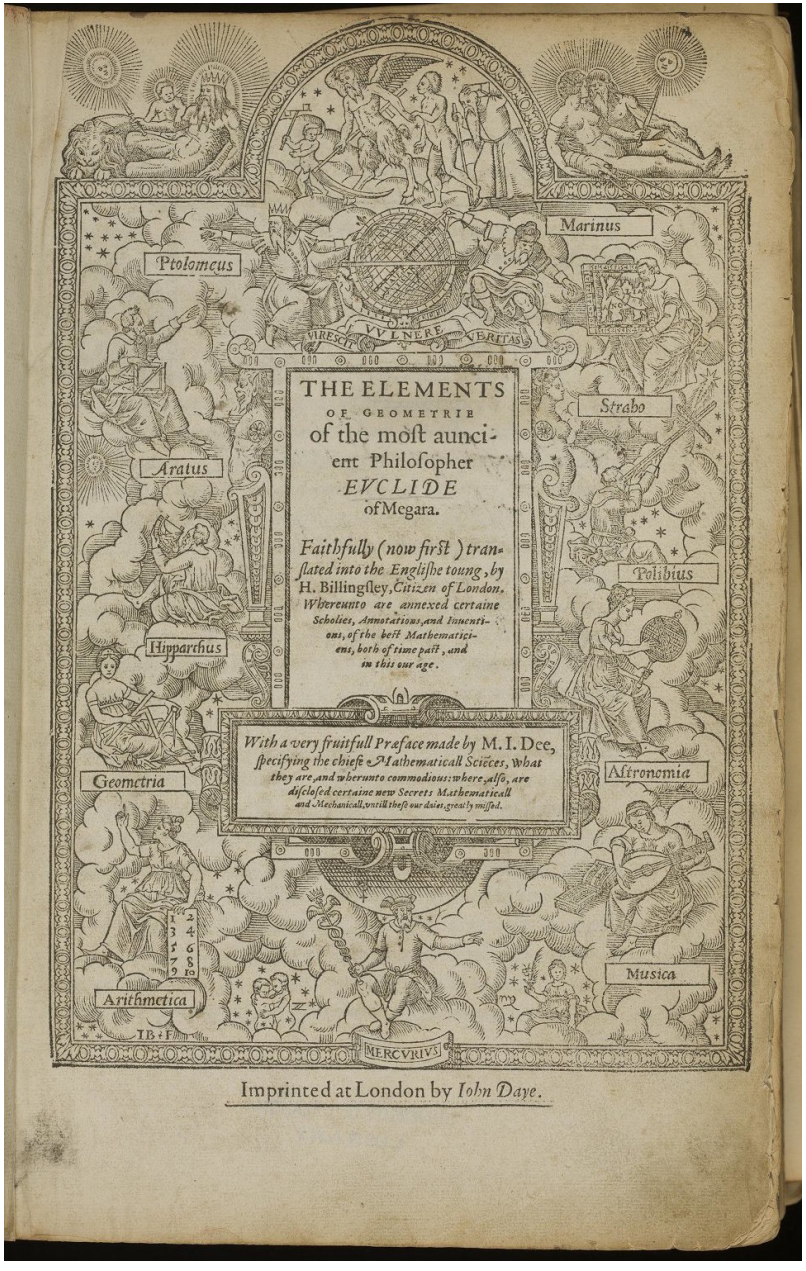
DEFINITIONS.

1. A **point** is that which has no part.
2. A **line** is breadthless length.
3. The extremities of a line are points.
4. A **straight line** is a line which lies evenly with the points on itself.
5. A **surface** is that which has length and breadth only.
6. The extremities of a surface are lines.
7. A **plane surface** is a surface which lies evenly with the straight lines on itself.
8. A **plane angle** is the inclination to one another of two lines in a plane which meet one another and do not lie in a straight line.
9. And when the lines containing the angle are straight, the angle is called **rectilineal.**
10. When a straight line set up on a straight line makes the adjacent angles equal to one another, each of the equal angles is **right,** and the straight line standing on the other is called a **perpendicular** to that on which it stands.
11. An **obtuse angle** is an angle greater than a right angle.
12. An **acute angle** is an angle less than a right angle.
13. A **boundary** is that which is an extremity of anything.
14. A **figure** is that which is contained by any boundary or boundaries.
15. A **circle** is a plane figure contained by one line such that all the straight lines falling upon it from one point among those lying within the figure are equal to one another ;

Canonical English edition by Sir Thomas L. Heath, 1908

See also the Reading Euclid Project

Billingsley's Euclid, 1570


Ptolomeus
Marinus
VIRESCIT VVLNERE VERITAS
THE ELEMENTS
OF GEOMETRIE
of the most auncient Philosopher
EVCLIDE
of Megara.
Faithfully (now first) translated into the Englishe toung, by H. Billingsley, Citizen of London.
Whereunto are annexed certaine Scholies, Annotations, and Inventions, of the best Mathematicians, both of time past, and in this our age.
With a very fruitfull Preface made by M. I. Dee, specifying the chiefe Mathematicall Sciences, what they are, and wherunto commodious: where, also, are disclosed certaine new Secrets Mathematicall and Mechanicall, until these our dayes greatly missed.
Aratus
Strabo
Polibius
Hipparchus
Geometria
Astronomia
Musica
Arithmetica
MERCVRIVS
Imprinted at London by Iohn Daye.

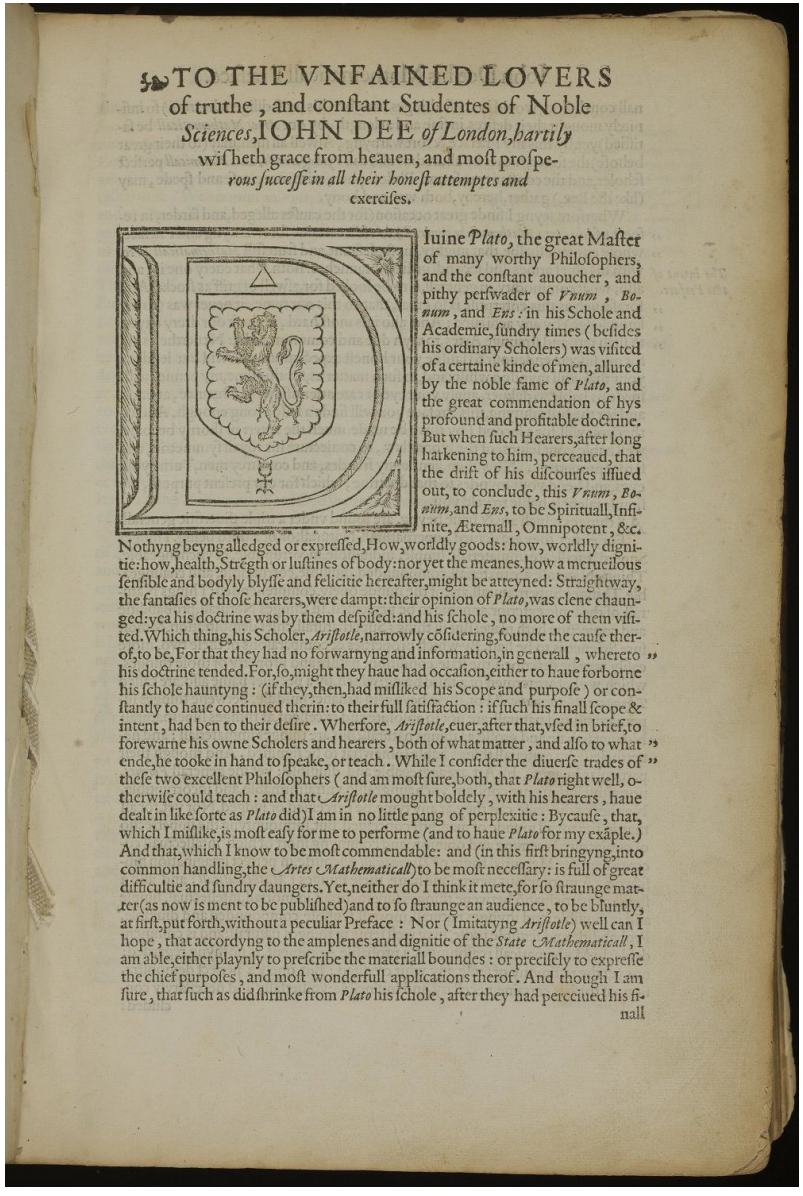
The Elements of Geometrie:

"Faithfully (now first) translated into the Englishe toung" by H. Billingsley, London, 1570

Available online

Preface by John Dee

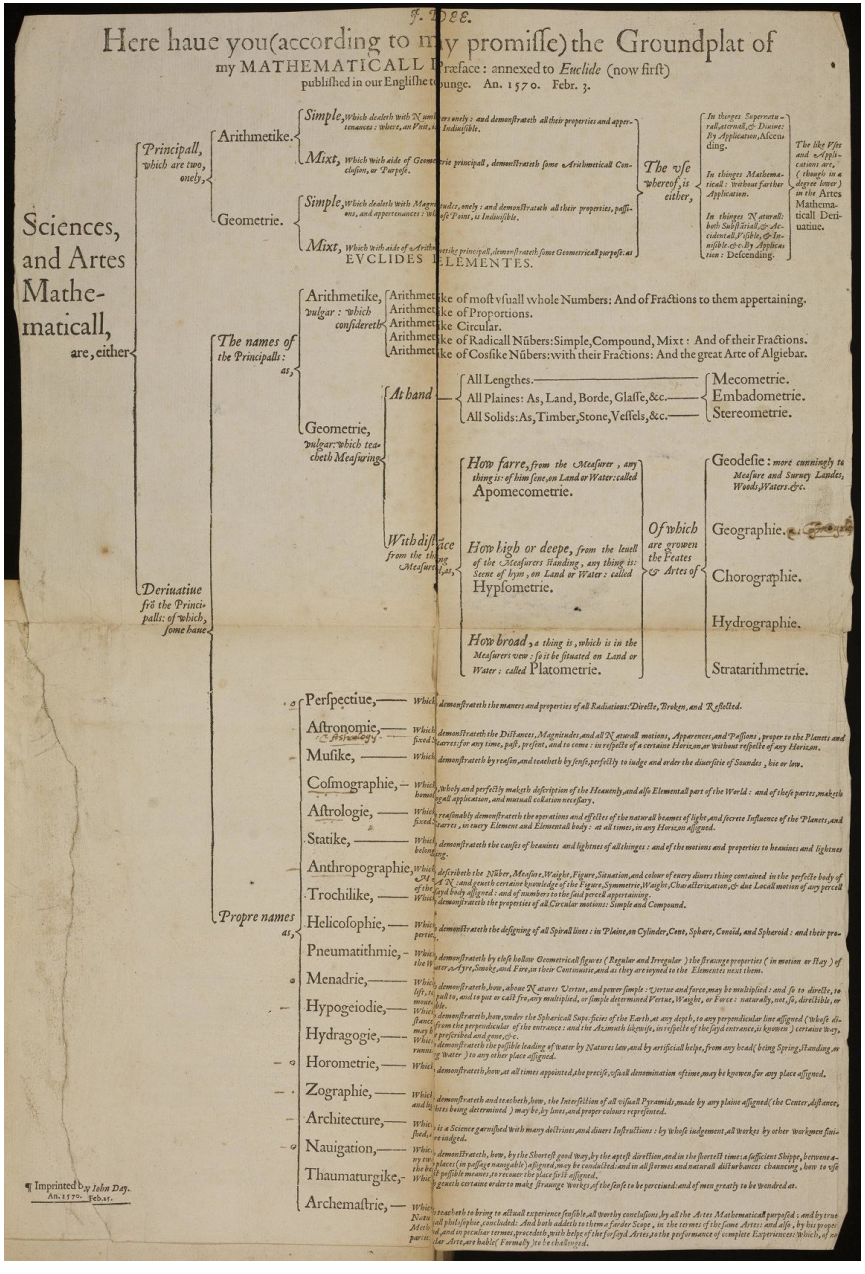
Dee's Preface



TO THE VNFAINED LOVERS of truthe, and constant Studentes of Noble *Sciences*, IOHN DEE *of London, hartily* wisheth grace from heauen, and most prospe- *rous succesſe in all their honeſt attemptes and* exerciſes.

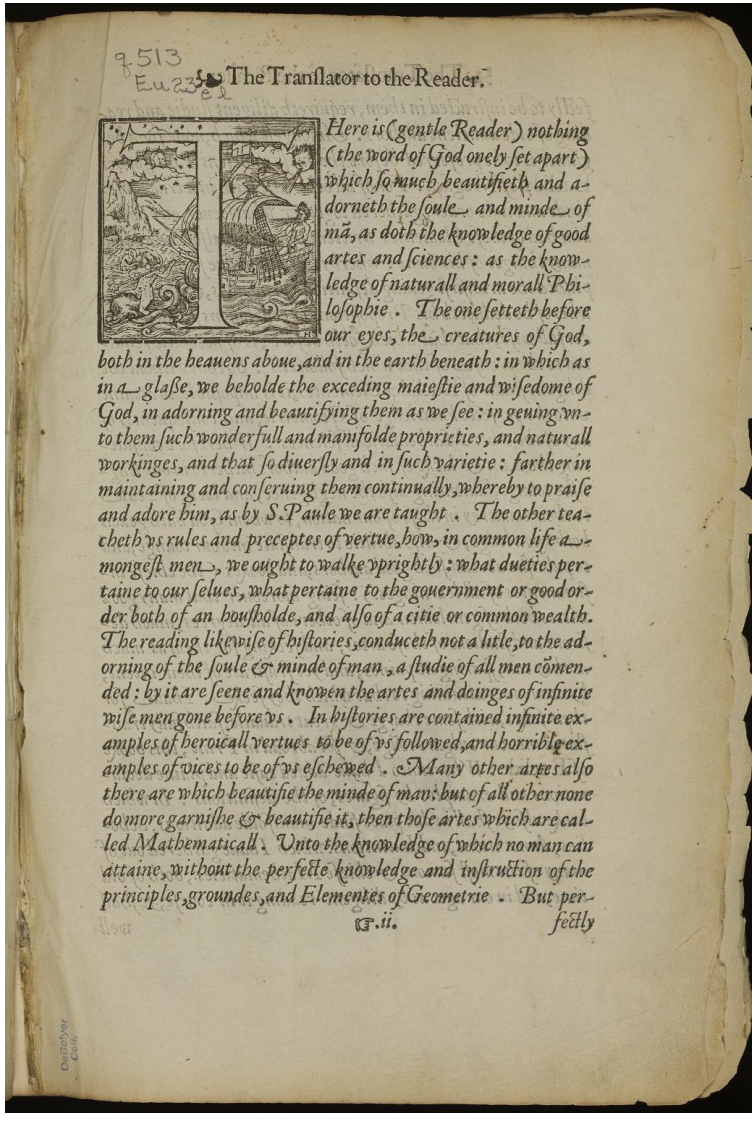
Iuine *Plato*, the great Master of many worthy Philosophers, and the constant auoucher, and pithy perswader of *Vnum*, *Bonum*, and *Ens*: in his Schole and Academie, sundry times (beſides his ordinary Scholers) was visited of a certaine kinde of men, allured by the noble fame of *Plato*, and the great commendation of hys profound and profitable doctrine. But when such Hearers, after long harkening to him, perceaued, that the drift of his discourses issued out, to conclude, this *Vnum*, *Bonum*, and *Ens*, to be Spirituall, Infinite, Æternall, Omnipotent, &c. Nothyng beyng alledged or expreſſed, How, worldly goods: how, worldly dignitie: how, health, Strēgth or luſtines of body: nor yet the meanes, how a merueilous ſenſible and bodyly blyſſe and felicitie hereafter, might be atteyned: Straightway, the fantaſies of thoſe hearers, were dampt: their opinion of *Plato*, was clene chaunged: yea his doctrine was by them deſpiſed: and his ſchole, no more of them viſited. Which thing, his Scholer, *Ariſtotle*, narrowly cōſidering, founde the cauſe therof, to be, For that they had no forwarnyng and information, in generall, whereto his doctrine tended. For, ſo, might they haue had occaſion, either to haue forborne his ſchole hauntyng: (if they, then, had miſliked his Scope and purpoſe) or conſtantly to haue continued therin: to their full ſatiſfaction: if ſuch his finall ſcope & intent, had ben to their deſire. Wherfore, *Ariſtotle*, euer, after that, vſed in brief, to forewarne his owne Scholers and hearers, both of what matter, and alſo to what ende, he tooke in hand to ſpeake, or teach. While I conſider the diuerſe trades of theſe two excellent Philoſophers (and am moſt ſure, both, that *Plato* right well, otherwiſe could teach: and that *Ariſtotle* mought boldely, with his hearers, haue dealt in like ſorte as *Plato* did) I am in no little pang of perplexitie: Bycauſe, that, which I miſlike, is moſt eaſy for me to performe (and to haue *Plato* for my exāple.) And that, which I know to be moſt commendable: and (in this firſt bringyng, into common handling, the *Artes Mathematicall*) to be moſt neceſſary: is full of great difficultie and ſundry daungers. Yet, neither do I think it mete, for ſo ſtraunge matter (as now is ment to be publiſhed) and to ſo ſtraunge an audience, to be bluntly, at firſt, put forth, without a peculiar Preface: Nor (Imitatyng *Ariſtotle*) well can I hope, that accordyng to the amplenes and dignitie of the *State Mathematicall*, I am able, either playnly to preſcribe the materiall boundes: or preciſely to expreſſe the chief purpoſes, and moſt wonderfull applications therof. And though I am ſure, that ſuch as did ſhrinke from *Plato* his ſchole, after they had perceiued his finall

Dee's 'Groundplat'


I. Dee.
Here haue you (according to my promisse) the Groundplat of
my MATHEMATICALL Præface: annexed to Euclide (now first)
published in our Englishe tounge. An. 1570. Febr. 3.
Sciences, and Artes Mathematicall, are, either
Principall, which are two, onely,
Arithmetike.
Geometrie.
Simple, Which dealeth with Numbers onely: and demonstrateth all their properties and appertenances: whether, as Point, is Indiuisible.
Mixt, Which with aide of Geometrie principall, demonstrateth some Arithmeticall Conclusion, or Purpose.
Simple, Which dealeth with Magnitudes, onely: and demonstrateth all their properties, passing as Point, is Indiuisible.
Mixt, Which with aide of Arithmetike principall, demonstrateth some Geometricall purpose: as
EVCLIDES ELEMENTES.
The vse whereof, is either,
In thinges Supernaturall, Astronall, & Diuine: By Application, Ascending.
In thinges Mathematicall: without further Application.
In thinges Naturall: both Sensible, & Accidentall, Visible, & Inuisible, & c. By Application: Descending.
The like Vse and Applications are, (though in a degree lower) in the Artes Mathematicall Deriuatiue.
The names of the Principalls: as,
Arithmetike, vulgar: which considereth
Arithmetike of most vsuall whole Numbers: And of Fractions to them appertaining.
Arithmetike of Proportions.
Arithmetike Circular.
Arithmetike of Radicall Nũbers: Simple, Compound, Mixt: And of their Fractions.
Arithmetike of Cosike Nũbers: with their Fractions: And the great Arte of Algiebar.
Geometrie, vulgar: which teacheth Measuring
At hand
All Lengthes. — Mecometrie.
All Plaines: As, Land, Borde, Glasse, &c. — Embadometrie.
All Solides: As, Timber, Stone, Vessels, &c. — Stereometrie.
With distance from the thing Measured
How farre, from the Measurer, any thing is: of him seene, on Land or Water: called Apomecometrie. — Geodesie: more cunningly to Measure and Survey Landes, Woods, Waters, &c.
How high or deepe, from the lowest of the Measurers Standing, any thing is: seene of him, on Land or Water: called Hypsometrie. — Geographie. Chorographie. Hydrographie.
How broad, a thing is, which is in the Measurers vew: so it be situated on Land or Water: called Platometrie. — Stratarithmetrie.
Of which are growen the Feates & Artes of
Deriuatiue fro the Principalls: of which, some haue
Propre names as,
Perspectiue, — Which demonstrateth the manners and properties of all Radiations: Direkte, Broken, and Reflected.
Astronomie, — Which demonstrateth the Distances, Magnitudes, and all Naturall motions, Apparences, and Passions, proper to the Planets and fixed Starres: for any time, past, present, and to come: in respecte of a certaine Horizon, or without respecte of any Horizon.
Musike, — Which demonstrateth by reason, and teacheth by sense, perfectly to iudge and order the diuerssitie of Soundes, hie or low.
Cosmographie, — Which wholy and perfectly maketh description of the Heauenly, and also Elementall part of the World: and of their parter, maketh mutuall application, and mutuall cohation necessary.
Astrologie, — Which reasonably demonstrateth the operations and effectes of the naturall beames of light, and secrete Influence of the Planets, and fixed Starres, to euery Elementall and Elementall body: at all times, to any Horizon assigned.
Statike, — Which demonstrateth the causes of heauines and lightnes of all thinges: and of the motions and properties to heauines and lightnes belonging.
Anthropographie, — Which describeth the Nũber, Measure, Waight, Figure, Situation, and colour of euery diuers thing contained in the perfecte body of Man: and exacte certaine knowledge of the Figure, Symmetrie, Waight, Characterization, & due Localle motion of any perfect sayd body assigned: and of numbers to the same perteining appertaining.
Trochilike, — Which demonstrateth the properties of all Circular motions: Simple and Compound.
Helicosophie, — Which demonstrateth the disgoining of all Spirall lines: in Plaine, in Cylinder, Cone, Sphere, Conoid, and Spheroid: and their properties.
Pneumatithmie, — Which demonstrateth by close hollow Geometricall figures (Regular and Irregular) the strange properties (in motion or stay) of the Water, Ayre, Smoke, and Fire, in their Continuitie, and as they are ioyned to the Elementes next them.
Menadrie, — Which demonstrateth, how, aboue Natures Vertue, and power simple: Vertue and force, may be multiplied: and so to direkte, to lift, to pull to, and to put or call fro, any multiplied, or simple determined Vertue, Waight, or Force: naturally, not so, direkible, or moueable.
Hypogeiodie, — Which demonstrateth, how, vnder the Spharicall Superficies of the Earth, at any depth, to any perpendicular line assigned (whose distance from the perpendicular of the entrance: and the Azimuth likewise, in respecte of the sayd entrance, is knowen) certaine way, may be prescribed and gone, &c.
Hydragogie, — Which demonstrateth the possible leading of water by Natures law, and by artificiall helpe, from any head (being Spring, Standing, or running water) to any other place assigned.
Horometrie, — Which demonstrateth, how, at all times appointed, the precise, or sayd denomination of time, may be knowen, for any place assigned.
Zographie, — Which demonstrateth and teacheth, how, the Intersition of all vsuall Pyramids, made by any plaine assigned (the Center, distances, and lights being determined) may be, by lines, and proper colours represented.
Architecture, — Which is a Science garnished with many delitable, and diuers Instrumentes: by whose iudgement, all workes by other workemen finished, are iudged.
Nauigation, — Which demonstrateth, how, by the Shortest good way, by the aptest direction, and in the shortest time: a sufficient Shippe, betweene places (on passage manyable) assigned, may be conducted: and in all stormes and naturall disturbances chaunsing, how to vse the possible meanes, to recouer the place first assigned.
Thaumaturgike, — Which teacheth certaine order to make strainge workes of the sense to be perceiued: and of men greatly to be wondred at.
Archemastrie, — Which teacheth to bring to all vsuall experience sensible, all worthy conclusions, by all the Artes Mathematicall purposed: and by true naturall philosophie concluded. And hath adderto to them a further Scope, in the termes of the same Artes: and also, by his proper rule, and in peculiar termes, proceedeth, with helpe of the forsayd Artes, to the performance of complete Experiences: Which, of no other Artes, are hable (Formally) to be challenged.
¶ Imprinted by John Day. An. 1570. Febr. 1.

See: Jennifer M. Rampling, 'The Elizabethan mathematics of everything: John Dee's 'Mathematicall praeface' to Euclid's *Elements*', *BSHM Bulletin: Journal of the British Society for the History of Mathematics* **26**(3) (2011) 135–146

Billingsley's Preface, pp. 1, 3



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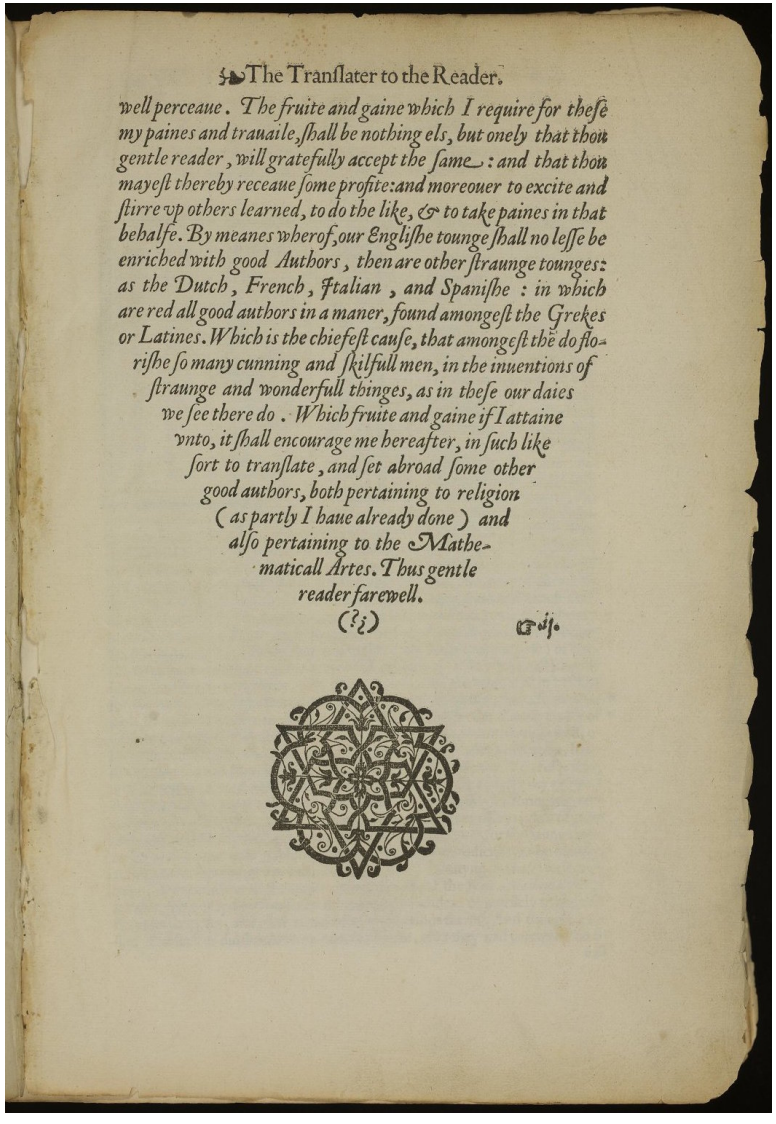
HEre is (gentle Reader) nothing (the word of God onely set apart) which so much beautifieth and adorneth the soule and minde of mã, as doth the knowledge of good artes and sciences: as the knowledge of naturall and morall Philosophie. The one setteth before our eyes, the creatures of God, both in the heauens aboue, and in the earth beneath: in which as in a glasse, we beholde the exceding maiestie and wisedome of God, in adorning and beautifying them as we see: in geuing vnto them such wonderfull and manifolde proprieties, and naturall workinges, and that so diuersly and in such varietie: farther in maintaining and conseruing them continually, whereby to praise and adore him, as by S. Paule we are taught. The other teacheth vs rules and preceptes of vertue, how, in common life amongest men, we ought to walke vprightly: what dueties pertaine to our selues, what pertaine to the gouernment or good order both of an housholde, and also of a citie or common wealth. The reading likewise of histories, conduceth not a litle, to the adorning of the soule & minde of man, a studie of all men cõmended: by it are seene and knowen the artes and doinges of infinite wise men gone before vs. In histories are contained infinite examples of heroicall vertues to be of vs followed, and horrible examples of vices to be of vs eschewed. Many other artes also there are which beautifie the minde of man: but of all other none do more garnishe & beautifie it, then those artes which are called Mathematicall. Vnto the knowledge of which no man can attaine, without the perfecte knowledge and instruction of the principles, groundes, and Elementes of Geometrie. But perfectly

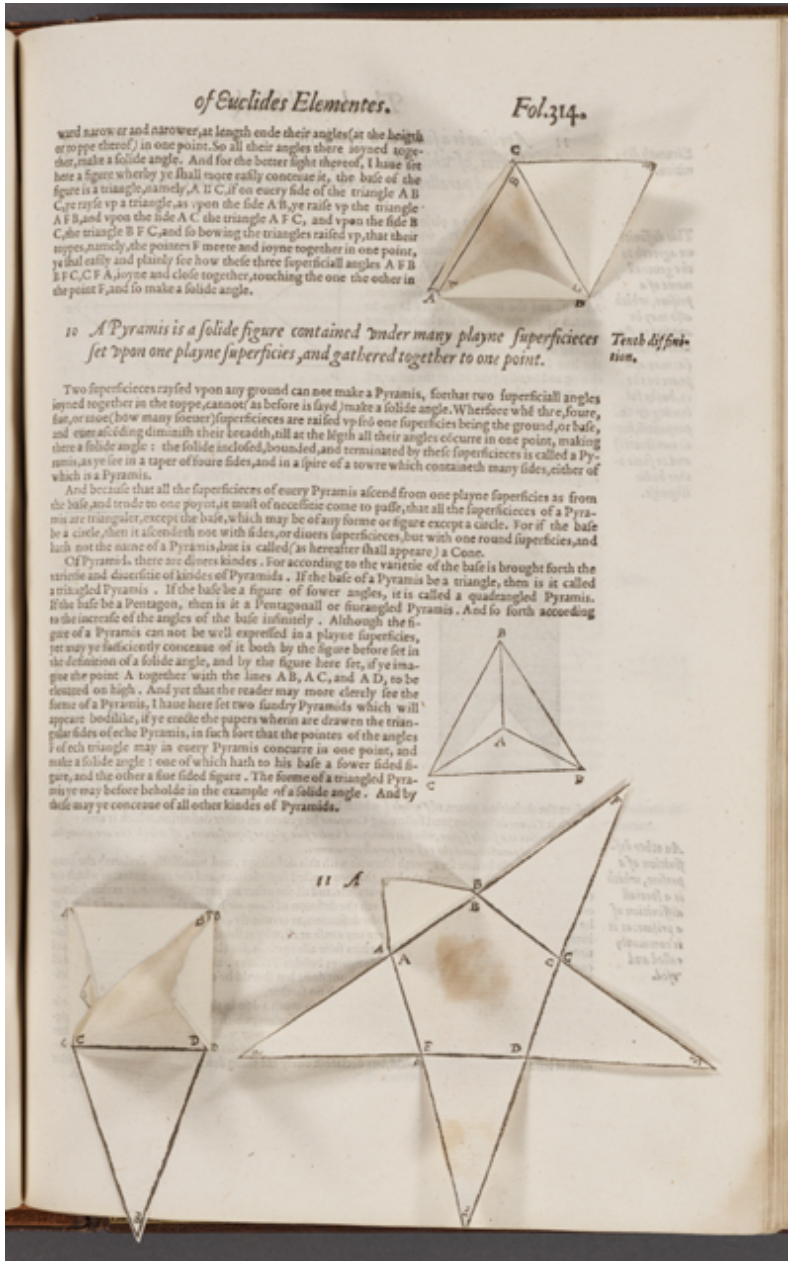
. . .



well perceaue. The fruite and gaine which I require for these my paines and trauaile, shall be nothing els, but onely that thou gentle reader, will gratefully accept the same: and that thou mayest thereby receaue some profite: and moreouer to excite and stirre vp others learned, to do the like, & to take paines in that behalfe. By meanes wherof, our Englishe tounge shall no lesse be enriched with good Authors, then are other straunge tounges: as the Dutch, French, Italian, and Spanishe: in which are red all good authors in a maner, found amongest the Grekes or Latines. Which is the chiefest cause, that amongest the do florishe so many cunning and skilfull men, in the inuentions of straunge and wonderfull thinges, as in these our daies we see there do. Which fruite and gaine if I attaine vnto, it shall encourage me hereafter, in such like sort to translate, and set abroad some other good authors, both pertaining to religion (as partly I haue already done) and also pertaining to the Mathematicall Artes. Thus gentle reader farewell.

(?i)

Pop-up Euclid



of Euclides Elementes. Fol.314.

ward narow er and narower, at length ende their angles (at the heigth or toppe therof) in one point. So all their angles there ioyned toge-ther, make a solide angle. And for the better light thereof, I haue set here a figure wherby ye shall more easily conceaue it, the base of the figure is a triangle, namely, A B C. If on euery side of the triangle A B C, ye rayse vp a triangle, as vpon the side A B, ye raise vp the triangle A F B, and vpon the side A C the triangle A F C, and vpon the side B C, the triangle B F C, and so bowing the triangles raised vp, that their toppes, namely, the pointes F meete and ioyne together in one point, ye shal easily and plainly see how these three superficiall angles A F B B F C, C F A, ioyne and close together, touching the one the other in the point F, and so make a solide angle.

10 *A Pyramis is a solide figure contained vnder many playne superficieces set vpon one playne superficies, and gathered together to one point.* *Tenth diffinition.*

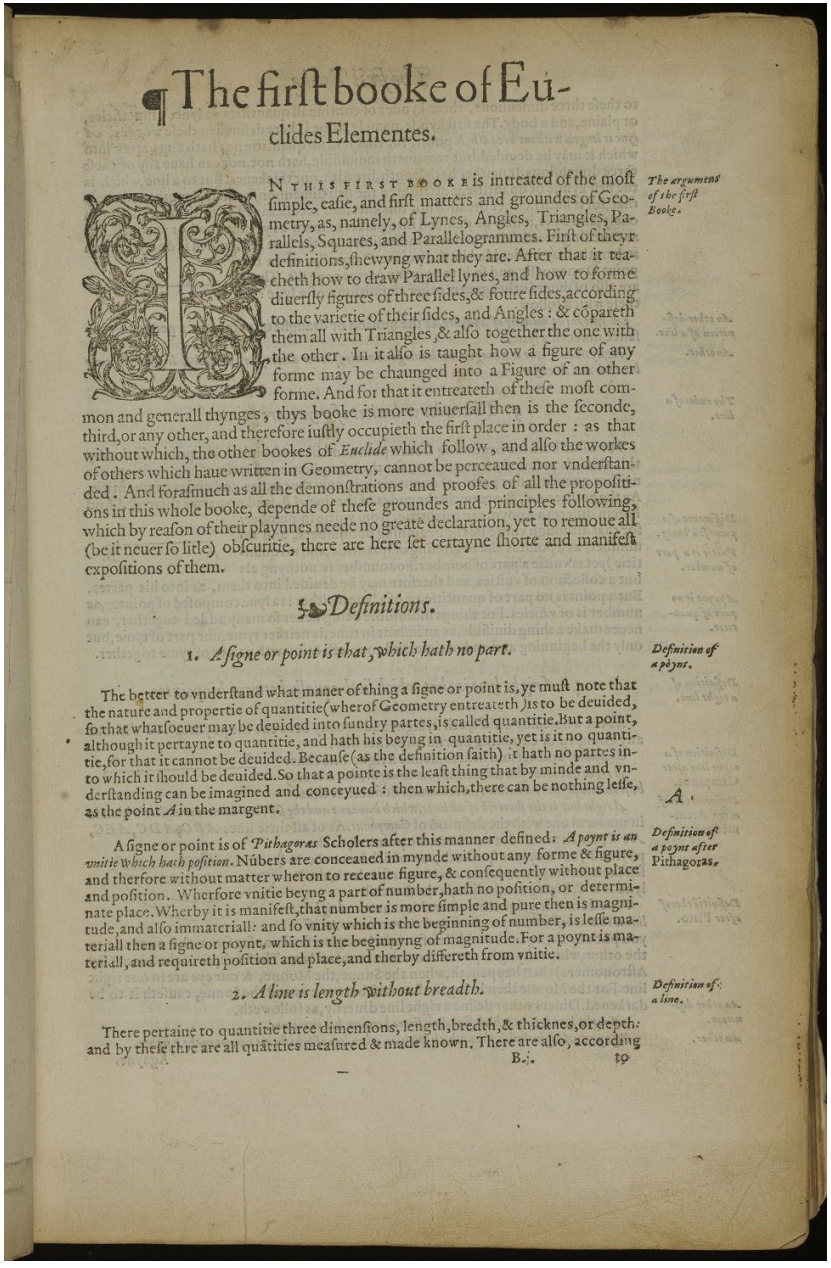
Two superficieces raysed vpon any ground can not make a Pyramis, forthat two superficiall angles ioyned together in the toppe, cannot (as before is sayd) make a solide angle. Wherfore whē three, foure, fiue, or more (how many soeuer) superficieces are raised vp frō one superficies being the ground, or base, and euer according diminish their breadth, till at the length all their angles ecclurre in one point, making there a solide angle: the solide inclosed, bounded, and terminated by these superficieces is called a Pyramis, as ye see in a taper of foure sides, and in a spire of a towre which containeth many sides, either of which is a Pyramis.

And because that all the superficieces of euery Pyramis ascend from one playne superficies as from the base, and tende to one poynt, it must of necessitie come to passe, that all the superficieces of a Pyramis are triangular, except the base, which may be of any forme or figure except a circle. For if the base be a circle, then it ascendeth not with sides, or diuers superficieces, but with one round superficies, and hath not the name of a Pyramis, but is called (as hereafter shall appeare) a Cone.

Of Pyramis there are diuers kindes. For according to the varietie of the base is brought forth the varietie and diuersitie of kindes of Pyramids. If the base of a Pyramis be a triangle, then is it called a triangled Pyramis. If the base be a figure of fower angles, it is called a quadrangled Pyramis. If the base be a Pentagon, then is it a Pentagonall or fiuerangled Pyramis. And so forth according to the increase of the angles of the base infinitely. Although the figure of a Pyramis can not be well expressed in a playne superficies, yet may ye sufficiently conceaue of it both by the figure before set in the definition of a solide angle, and by the figure here set, if ye imagine the point A together with the lines A B, A C, and A D, to be erected on high. And yet that the reader may more clerely see the forme of a Pyramis, I haue here set two sundry Pyramids which will appeare bodilike, if ye encise the papers wherin are drawen the triangular sides of eche Pyramis, in such sort that the pointes of the angles of eech triangle may in euery Pyramis concurre in one point, and make a solide angle: one of which hath to his base a fower sided figure, and the other a fiue sided figure. The forme of a triangled Pyramis ye may before beholde in the example of a solide angle. And by these may ye conceaue of all other kindes of Pyramids.

11 *A*

Book I: definitions



¶ The first booke of Euclides Elementes.

The argument of the first Booke.

THIS FIRST BOOKE is intreated of the most simple, easie, and first matters and grounde of Geometry, as, namely, of Lynes, Angles, Triangles, Parallels, Squares, and Parallelogrammes. First of theyr definitions, shewyng what they are. After that it teacheth how to draw Parallel lynes, and how to forme diuersly figures of three sides, & foure sides, according to the varietie of their sides, and Angles: & cõpareth them all with Triangles, & also together the one with the other. In it also is taught how a figure of any forme may be chaunged into a Figure of an other forme. And for that it entreateth of these most common and generall thynges, thys booke is more vniuersall then is the seconde, third, or any other, and therefore iustly occupieth the first place in order: as that without which, the other bookes of *Euclide* which follow, and also the workes of others which haue written in Geometry, cannot be perceaued nor vnderstanded. And forasmuch as all the demonstrations and proofes of all the propositions in this whole booke, depende of these groundes and principles following, which by reason of their playnnes neede no greate declaration, yet to remoue all (be it neuer so litle) obscuritie, there are here set certayne shorte and manifest expositions of them.

Definitions.

Definition of a poyns.

1. *A signe or point is that, which hath no part.*

The better to vnderstand what maner of thing a signe or point is, ye must note that the nature and propertie of quantitie (wherof Geometry entreateth) is to be deuided, so that whatsoeuer may be deuided into sundry partes, is called quantitie. But a point, although it pertayne to quantitie, and hath his beyng in quantitie, yet is it no quantitie, for that it cannot be deuided. Because (as the definition saith) it hath no partes into which it should be deuided. So that a pointe is the least thing that by minde and vnderstanding can be imagined and conceyued: then which, there can be nothing lesse, as the point *A* in the margent.

A

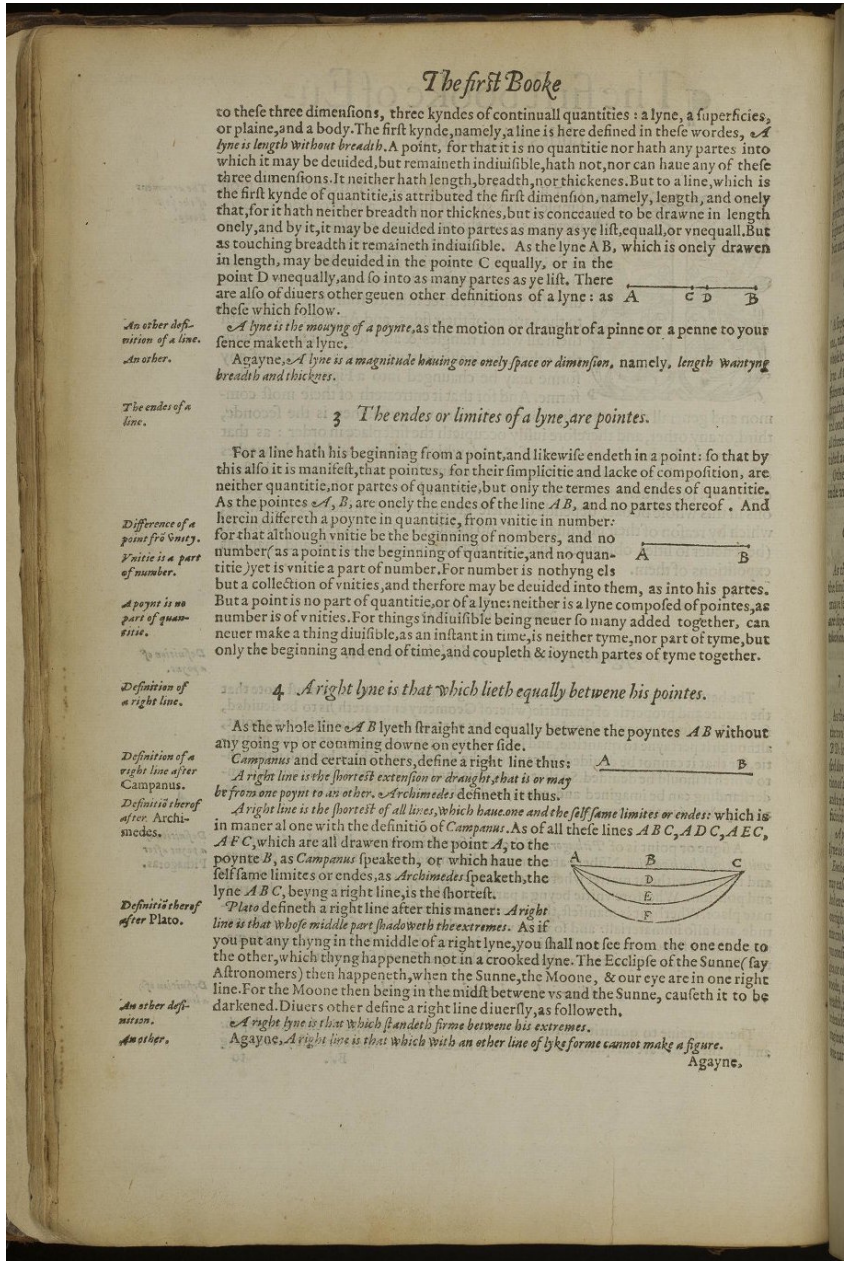
Definition of a poynt after Pithagoras.

A signe or point is of *Pithagoras* Scholers after this manner defined: *A poynt is an vnitie which hath position.* Nũbers are conceaued in mynde without any forme & figure, and therfore without matter wherõto receaue figure, & consequently without place and position. Wherfore vnitie beyng a part of number, hath no position, or determinate place. Wherby it is manifest, that number is more simple and pure then is magnitude, and also immateriall: and so vnity which is the beginning of number, is lesse materiall then a signe or poynt, which is the beginnyng of magnitude. For a poynt is materiall, and requireth position and place, and therby differeth from vnitie.

Definition of a line.

2. *A line is length without breadth.*

There pertaine to quantitie three dimensions, length, bredth, & thicknes, or depth: and by these thre are all quãtities measured & made known. There are also, according



to these three dimensions, three kyndes of continuall quantities: a lyne, a superficies, or plaine, and a body. The first kynde, namely, a line is here defined in these wordes, *A lyne is length without breadth.* A point, for that it is no quantitie nor hath any partes into which it may be deuided, but remaineth indiuisible, hath not, nor can haue any of these three dimensions. It neither hath length, breadth, nor thickenes. But to a line, which is the first kynde of quantitie, is attributed the first dimension, namely, length, and onely that, for it hath neither breadth nor thicknes, but is conceaued to be drawne in length onely, and by it, it may be deuided into partes as many as ye list, equall, or vnequall. But as touching breadth it remaineth indiuisible. As the lyne A B, which is onely drawen in length, may be deuided in the pointe C equally, or in the point D vnequally, and so into as many partes as ye list. There are also of diuers other geuen other definitions of a lyne: as these which follow.

A C D B

An other definition of a line.

A lyne is the mouyng of a poynte, as the motion or draught of a pinne or a penne to your sence maketh a lyne.

An other.

Agayne, *A lyne is a magnitude hauing one onely space or dimension,* namely, *length wantyng breadth and thicknes.*

The endes of a line.

3 *The endes or limites of a lyne, are pointes.*

For a line hath his beginning from a point, and likewise endeth in a point: so that by this also it is manifest, that pointes, for their simplicitie and lacke of composition, are neither quantitie, nor partes of quantitie, but only the termes and endes of quantitie. As the pointes *A, B,* are onely the endes of the line *A B,* and no partes thereof. And herein differeth a poynte in quantitie, from vnitie in number: for that although vnitie be the beginning of nombers, and no number (as a point is the beginning of quantitie, and no quantitie) yet is vnitie a part of number. For number is nothyng els but a collection of vnities, and therfore may be deuided into them, as into his partes. But a point is no part of quantitie, or of a lyne: neither is a lyne composed of pointes, as number is of vnities. For things indiuisible being neuer so many added together, can neuer make a thing diuisible, as an instant in time, is neither tyme, nor part of tyme, but only the beginning and end of time, and coupleth & ioyneth partes of tyme together.

Difference of a point frõ vnity.

Vnitie is a part of number.

A B

A poynt is no part of quantitie.

Definition of a right line.

4 *A right lyne is that which lieth equally betwene his pointes.*

As the whole line *A B* lyeth straight and equally betwene the poyntes *A B* without any going vp or comming downe on eyther side.

Definition of a right line after Campanus.

Campanus and certain others, define a right line thus: *A right line is the shortest extension or draught, that is or may be from one poynt to an other.*

A B

Definitiõ thereof after Archimedes.

Archimedes defineth it thus. *A right line is the shortest of all lines, which haue one and the self same limites or endes:* which is in maner al one with the definitiõ of *Campanus.* As of all these lines *A B C, A D C, A E C, A F C,* which are all drawen from the point *A,* to the poynte *B,* as *Campanus* speaketh, or which haue the self same limites or endes, as *Archimedes* speaketh, the lyne *A B C,* beyng a right line, is the shortest.

A B C D E F

Definitiõ thereof after Plato.

Plato defineth a right line after this maner: *A right line is that whose middle part shadoweth the extremes.* As if you put any thyng in the middle of a right lyne, you shall not see from the one ende to the other, which thyng happeneth not in a crooked lyne. The Eccliפse of the Sunne (say Astronomers) then happeneth, when the Sunne, the Moone, & our eye are in one right line. For the Moone then being in the midst betwene vs and the Sunne, causeth it to be darkened. Diuers other define a right line diuersly, as followeth.

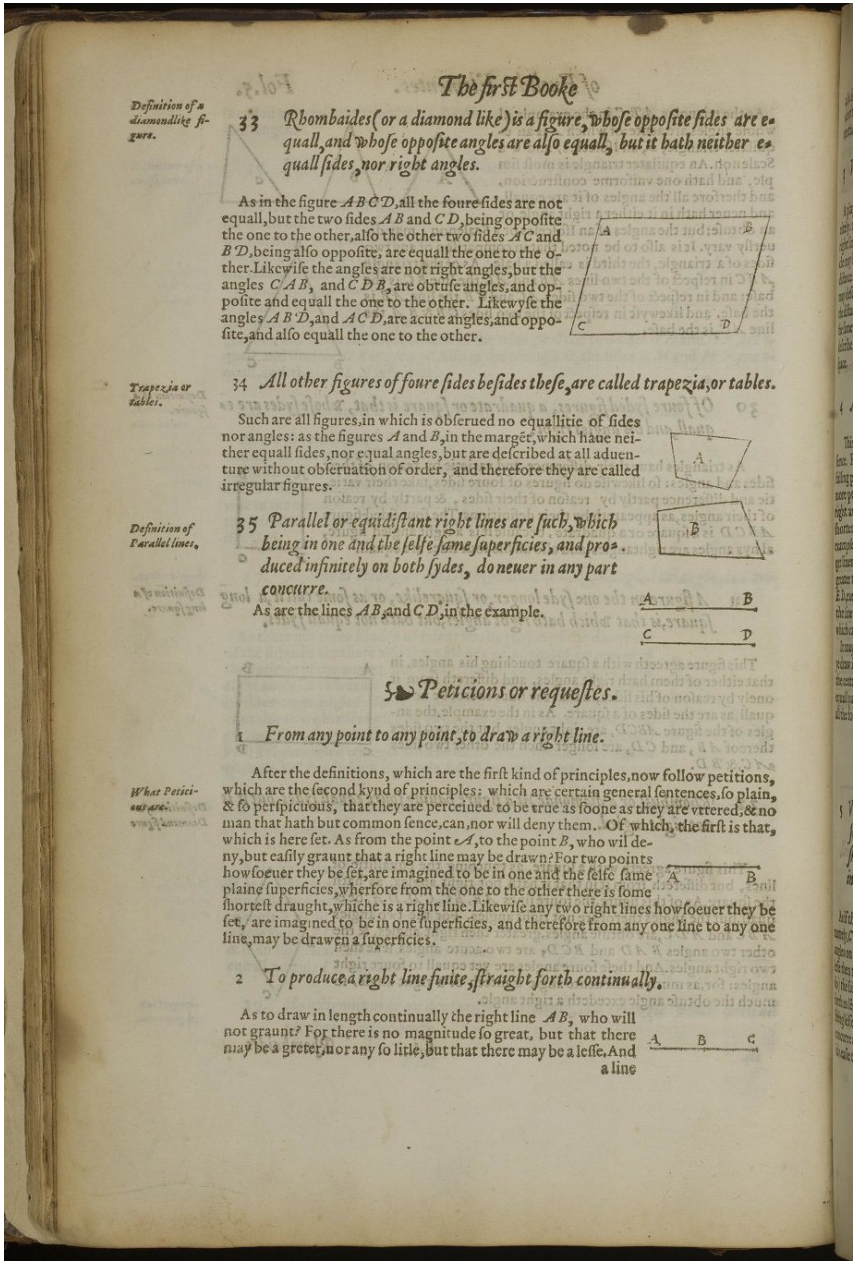
An other definition.

A right lyne is that which standeth firme betwene his extremes.

An other.

Agayne, *A right line is that which with an other line of lyke forme cannot make a figure.*

Book I: postulates



Definition of a diamondlike figure.

33 *Rhombaides (or a diamond like) is a figure, whose oppositte sides are equall, and whose oppositte angles are also equall, but it hath neither equall sides, nor right angles.*

As in the figure *A B C D*, all the foure sides are not equall, but the two sides *A B* and *C D*, being oppositte the one to the other, also the other two sides *A C* and *B D*, being also oppositte, are equall the one to the other. Likewise the angles are not right angles, but the angles *C A B*, and *C D B*, are obtuse angles, and oppositte and equall the one to the other. Likewyse the angles *A B D*, and *A C D*, are acute angles, and oppositte, and also equall the one to the other.

Trapezia or tables.

34 *All other figures of foure sides besides these, are called trapezia, or tables.*

Such are all figures, in which is obserued no equallitie of sides nor angles: as the figures *A* and *B*, in the marget, which haue neither equall sides, nor equal angles, but are described at all aduenture without obseruation of order, and therefore they are called irregular figures.

Definition of Parallel lines.

35 *Parallel or equidistant right lines are such, which being in one and the selfe same superficies, and produced infinitely on both sydes, do neuer in any part concurre.*

As are the lines *A B*, and *C D*, in the example.

Peticions or requestes.

1 *From any point to any point, to draw a right line.*

What Peticions are.

After the definitions, which are the first kind of principles, now follow petitions, which are the second kynd of principles: which are certain general sentences, so plain, & so perspicuous, that they are perceiued to be true as soone as they are vttered, & no man that hath but common sence, can, nor will deny them. Of which, the first is that, which is here set. As from the point *A*, to the point *B*, who wil deny, but easily graunt that a right line may be drawn? For two points howsoeuer they be set, are imagined to be in one and the selfe same plaine superficies, wherfore from the one to the other there is some

shortest draught, whiche is a right line. Likewise any two right lines howsoeuer they be set, are imagined to be in one superficies, and therefore from any one line to any one line, may be drawen a superficies.

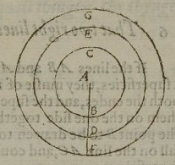
2 *To produce a right line finite, straight forth continually.*

As to draw in length continually the right line *A B*, who will not graunt? For there is no magnitude so great, but that there may be a greter, no or any so litle, but that there may be a lesse. And a line

a line is a draught from one point to an other, therfore from the point *B*, which is the ende of the line *A B*, may be drawn a line to some other point, as to the point *C*, and from that to an other, and so infinitely.

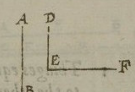
3 *Vpon any centre and at any distance, to describe a circle.*

A playne superficies may in compasse be extended infinitely: as from any pointe to any pointe may be drawen a right line, by reason wherof it commeth to passe that a circle may be described vpon any centre and at any space or distance. As vpon the centre *A*, and vpon the space *A B*, ye may describe the circle *B C*, & vpon the same centre, vpon the distance *A D*, ye may describe the circle *D E*, or vpon the same centre *A*, according to the distaunce *A F*, ye may describe the circle *F G*, and so infinitely extendyng your space.

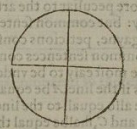


4 *All right angles are equall the one to the other.*

This peticion is most plaine, and offreth it selfe euen to the sence. For as much as a right angle is caused of one right lyne falling perpendicularly vppon an other, and no one line can fall more perpendicularly vpo a line then an other: therfore no one right angle can be greater thē an other: neither do the length or shortenes of the lines alter the greatnes of the angle. For in the example, the right angle *A B C*, though it be made of much longer lines then the right angle *D E F*, whose lines are much shorter, yet is that angle no greater then the other. For if ye set the point *E* iust vpon the point *B*, then shal the line *E D*, euenly and iustly fall vpon the line *A B*, and the line *E F*, shall also fall equally vpon the line *B C*, and so shal the angle *D E F*, be equall to the angle *A B C*, for that the lines which cause them, are of like inclination.

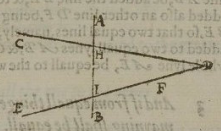


It may euidently also be sene at the centre of a circle. For if ye draw in a circle two diameters, the one cutting the other in the centre by right angles, ye shall deuide the circle into fowre equall partes, of which eche contayneth one right angle, so are all the foure right angles about the centre of the circle equall.



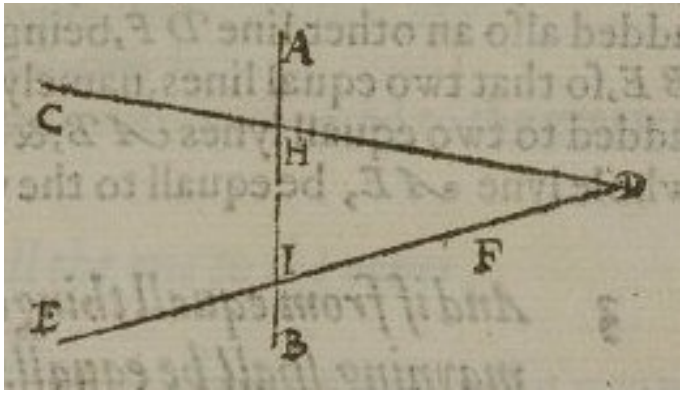
5 *VVhen a right line falling vpon two right lines, doth make on one & the selfe same syde, the two inwarde angles lesse then two right angles, then shal these two right lines beyng produced at length concurre on that part, in which are the two angles lesse then two right angles.*

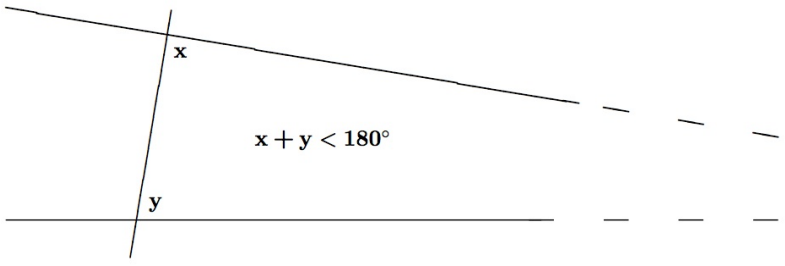
As if the right line *A B*, fall vpon two right lines, namely, *C D* and *E F*, so that it make the two inward angles on the one side, as the angles *D H I* & *F I H*, lesse then two right angles (as in the example they do) the said two lines *C D*, and *E F*, being drawen forth in legth on that part, wheron the two angles being lesse the two right angles consist, shal at legth concurre and meete together: as in the point *D*, as it is easie to see. For the partes of the lines towardes *D F*, are more enclined the one to



Postulate 5

5 *VVhen a right line falling vpon two right lines, doth make on one & the selfe same syde, the two inwarde angles leſse then two right angles, then ſhal theſe two right lines beyng produced at length concurre on that part, in which are the two angles leſse then two right angles.*





Equivalent formulation (Proclus, 5th century; John Playfair, 1795): given a straight line L and a point P not on L there is one and only one straight line through P that is parallel to L.

Classical disquiet about the fifth postulate

Original to Euclid? Less 'self-evident' than the other postulates?

Euclid used it (e.g., in the proof of Proposition 29 of Book I), so the property is necessary — but does it in fact follow from the other postulates?

Proclus in commentary on Euclid, 5th century (after citing Ptolemy's attempted proof of the parallel postulate, and discussing the nature of truth, with reference to Aristotle and Plato):

> *It is then clear from this that we must seek a proof of the present theorem, and that it is alien to the special character of postulates.*

Attempted (unsuccessfully) to prove the fifth postulate on the basis of the others

See Heath, pp. 202–220

Mediaeval disquiet about the fifth postulate

In the Islamic world:

Ibn al-Haytham (Alhazen) (965–1039) attempted (unsuccessfully) to prove the parallel postulate by contradiction

Omar Khayyám (1050–1123) attempted to prove the fifth postulate on the basis of the following alternative:

> *two convergent straight lines intersect and it is impossible for two convergent straight lines to diverge in the direction in which they converge*

Described the situations that may occur if the postulate is omitted

Nasir al-Din al-Tusi (1201–1274) criticised Khayyám's attempted proof, offered his own

Al-Tusi's thoughts found their way into Europe via the writings (1298) of his son Sadr al-Tusi

Early modern disquiet about the fifth postulate

After reading al-Tusi, John Wallis showed that the parallel postulate is equivalent to the following:

> *on a given finite straight line it is always possible to construct a triangle similar to a given triangle*

He lectured on this in Oxford in 1663

Attempts to prove the fifth postulate on the basis of Euclid's other axioms had resulted only in equivalent forms — so can we have a consistent geometry in which it the parallel postulate fails?

Early hints of non-Euclidean geometry

Giovanni Girolamo Saccheri (1667–1733): sought to establish the validity of Euclidean geometry — negated the parallel postulate in search of a contradiction; two cases:

- internal angles of a triangle add up to less than two right angles — contradicts Euclid's second postulate
- internal angles of a triangle add up to more than two right angles — leads to non-intuitive ideas

Similar results derived by Johann Heinrich Lambert (1728–1777) in his *Theorie der Parallellinien* (1766)

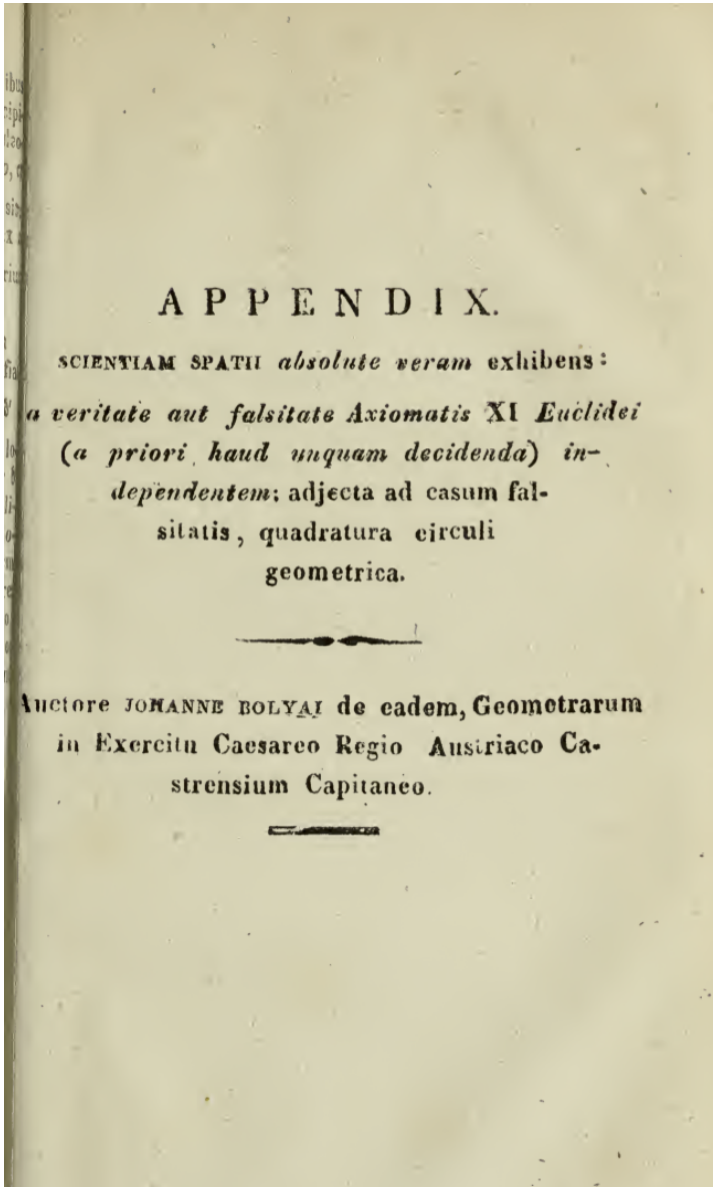
Non-Euclidean geometries

Consistent non-Euclidean geometry probably first constructed (tentatively) by Gauss, c. 1817–1830, but remained unpublished

Problem pursued independently (without success) by Gauss' friend Farkas Bolyai (1775–1856)

Pursued (against paternal advice) and solved by János Bolyai (1802–1860): "I have created a new and different world out of nothing" (1823)

Bolyai's geometry



APPENDIX.

SCIENTIAM SPATII *absolute veram* exhibens:

a veritate aut falsitate Axiomatis XI *Euclidei* (*a priori haud unquam decidenda*) *independentem*; adjecta ad casum falsitatis, quadratura circuli geometrica.

Auctore JOHANNE BOLYAI de eadem, Geometrarum in Exercitu Caesareo Regio Austriaco Castrensium Capitaneo.

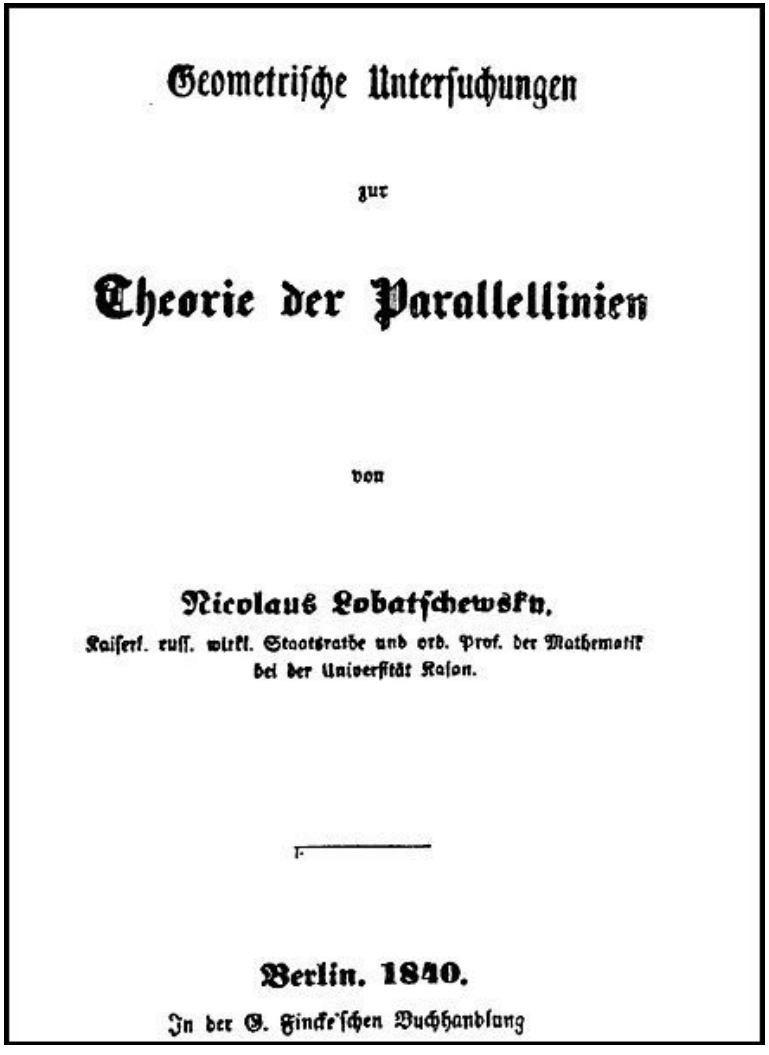
Published as appendix 'The science absolute of space: independent of the truth or falsity of Euclid's axiom XI (which can never be decided a priori)' to father's textbook *Tentamen iuventutem studiosam in elementa matheosos introducendi* (1832)

English translation by George Bruce Halstead (1896)

Meanwhile in Russia...

Non-Euclidean geometry developed independently by Nikolai Ivanovich Lobachevskii [Николай Иванович Лобачевский] (1792–1856) using the negation of Playfair's axiom

Lobachevskii's works



Geometriſche Unterſuchungen

zur

Theorie der Parallellinien

von

Nicolaus Lobatſchewsky,

Kaiſerl. ruſſ. wirkl. Staatsrathe und ord. Prof. der Mathematik bei der Univerſität Kaſan.

Berlin. 1840.

In der G. Fincke'ſchen Buchhandlung

Complicated story of dissemination...

Geometriya [Геометрия] written in 1823 but not published until 1909

Ideas presented in Kazan in 1826, published there 1829 — but rejected by St Petersburg Academy

Other works in Russian, French and German, including *Geometrische Untersuchungen zur Theorie der Parallellinien* (1840), *Pangéométrie* (1855)

(See Tom Lehrer for an unfair characterisation of Lobachevskii: https://youtu.be/IL4vWJbwmqM)

Acceptance and impact of non-Euclidean geometries

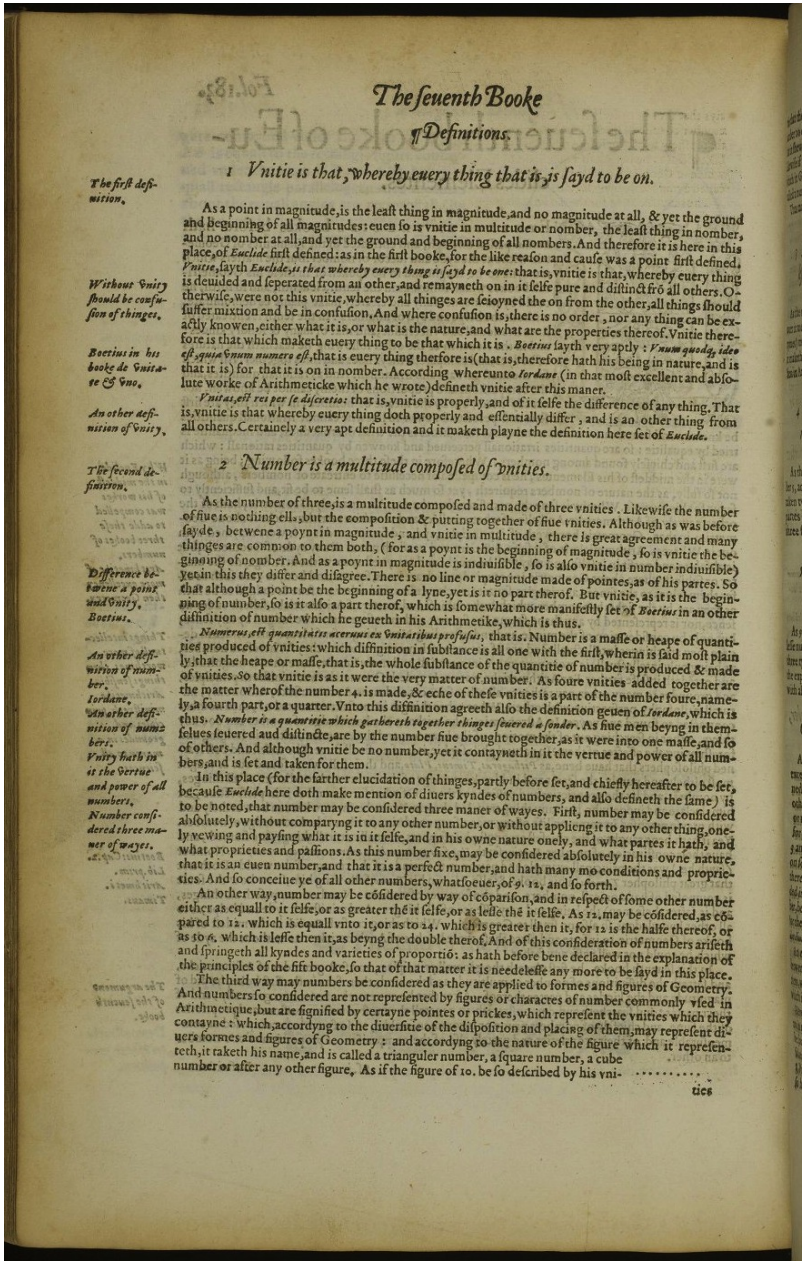
Slow to gain acceptance due to

- obscurity of publications
- lack of intuitive understanding

But non-Euclidean geometries

- overturned old ideas of mathematical certainty
- introduced new ideas about space
- helped drive the late 19th-century move towards axiomatisation

Euclid on numbers (positive integers)



The seuenth Booke

¶ Definitions.

The first definition.

1 *Vnitie is that, whereby euery thing that is, is sayd to be on.*

As a point in magnitude, is the least thing in magnitude, and no magnitude at all, & yet the ground and beginning of all magnitudes: euen so is vnitie in multitude or nomber, the least thing in nomber, and no nomber at all, and yet the ground and beginning of all nombers. And therefore it is here in this place, of *Euclide* first defined: as in the first booke, for the like reason and cause was a point first defined. *Vnitie*, sayth *Euclide, is that whereby euery thing is sayd to be one*: that is, vnitie is that, whereby euery thing is deuided and seperated from an other, and remayneth on in it selfe pure and distinct frō all others. Otherwise, were not this vnitie, whereby all thinges are seioyned the on from the other, all things should suffer mixtion and be in confusion. And where confusion is, there is no order, nor any thing can be exactly knowen, either what it is, or what is the nature, and what are the properties thereof. Vnitie therefore is that which maketh euery thing to be that which it is. *Boetius* sayth very aptly: *Vnum quodq; ideo est, quia vnum numero est*, that is euery thing therfore is (that is, therefore hath his being in nature, and is that it is) for that it is on in nomber. According whereunto *Iordane* (in that most excellent and absolute worke of Arithmeticke which he wrote) defineth vnitie after this maner.

Vnitas, est res per se discretio: that is, vnitie is properly, and of it selfe the difference of any thing. That is, vnitie is that whereby euery thing doth properly and essentially differ, and is an other thing from all others. Certainely a very apt definition and it maketh playne the definition here set of *Euclide*.

Without vnitie should be confusion of thinges.

Boetius in his booke de vnitate & vno.

An other definition of vnitie.

The second definition.

2 *Number is a multitude composed of vnities.*

As the number of three, is a multitude composed and made of three vnities. Likewise the number of fiue is nothing ell, but the composition & putting together of fiue vnities. Although as was before sayde, betwene a poynt in magnitude, and vnitie in multitude, there is great agreement and many thinges are common to them both, (for as a poynt is the beginning of magnitude, so is vnitie the beginning of nomber. And as a poynt in magnitude is indiuisible, so is also vnitie in number indiuisible) yet in this they differ and disagree. There is no line or magnitude made of pointes, as of his partes. So that although a point be the beginning of a lyne, yet is it no part therof. But vnitie, as it is the beginning of number, so is it also a part therof, which is somewhat more manifestly set of *Boetius* in an other diffinition of number which he geueth in his Arithmetike, which is thus.

Numerus, est quantitates ocurrens ex vnitatibus profusus, that is. Number is a masse or heape of quantities produced of vnities: which diffinition in substance is all one with the first, wherin is said most plainly, that the heape or masse, that is, the whole substance of the quantitie of number is produced & made of vnities. So that vnitie is as it were the very matter of number. As foure vnities added together are the matter wherof the number 4. is made, & eche of these vnities is a part of the number foure, namely, a fourth part, or a quarter. Vnto this diffinition agreeth also the definition geuen of *Iordane*, which is thus. *Number is a quantitie which gethereth together thinges seuered a sonder.* As fiue men beyng in themselues seuered and distincte, are by the number fiue brought together, as it were into one masse, and so of others. And although vnitie be no number, yet it contayneth in it the vertue and power of all nombers, and is set and taken for them.

Difference betwene a point and vnity. Boetius.

An other definition of number. Iordane.

An other definition of numbers. Vnity hath in it the vertue and power of all numbers.

Number considered three maner of wayes.

In this place (for the farther elucidation of thinges, partly before set, and chiefly hereafter to be set, because *Euclide* here doth make mention of diuers kyndes of numbers, and also defineth the same) is to be noted, that number may be considered three maner of wayes. First, number may be considered absolutely, without comparyng it to any other number, or without applieng it to any other thing, onely vewing and paysing what it is in it selfe, and in his owne nature onely, and what partes it hath, and what proprieties and passions. As this number fiue, may be considered absolutely in his owne nature, that it is an euen number, and that it is a perfect number, and hath many mo conditions and proprieties. And so conceiue ye of all other numbers, whatsoeuer, of 9. 12, and so forth.

An other way, number may be cōsidered by way of cōparison, and in respect of some other number either as equall to it selfe, or as greater thē it selfe, or as lesse thē it selfe. As 12, may be cōsidered, as cōpared to 12. which is equall vnto it, or as to 24. which is greater then it, for 12 is the halfe thereof, or as to 6. which is lesse then it, as beyng the double therof. And of this consideration of numbers ariseth and springeth all kyndes and varieties of proportiō: as hath before bene declared in the explanation of the principles of the fift booke, so that of that matter it is needeleese any more to be sayd in this place.

The third way may numbers be considered as they are applied to formes and figures of Geometry. And numbers so considered are not represented by figures or characters of number commonly vsed in Arithmetique, but are signified by certayne pointes or prickes, which represent the vnities which they contayne: which, accordyng to the diuersitie of the disposition and placing of them, may represent diuers formes and figures of Geometry: and accordyng to the nature of the figure which it representeth, it taketh his name, and is called a trianguler number, a square number, a cube number or after any other figure. As if the figure of 10. be so described by his vnities that they be orderly set in a straight course, so that they represent the forme of a line, then is the number 10 called a lineall number. And if the same number 10 be so described by his vnities, that it shew forth the forme of a triangle, then it is called a trianguler nūber: as ye here see. Likewise if 12 be in such sort described by his vnities, that it representeth that forme or figure which in Geometry is called a figure on the one side longer: then shall the number 12 be called a number hauyng the one side longer, and so may you conceaue of all others.

Thus much of this for the declaration of the thinges following.

The third definition.

3 *A part is a lesse number in comparison to the greater when the lesse measureth the greater.*

As the number 3 compared to the number 12. is a part. For 3 is a lesse number then is 12. and moreouer it measureth 12 the greater number. For 3 taken (or added to it selfe) certayne times (namely, 4 tymes) maketh 12. For 3 foure tymes is 12. Likewise is 2 a part of 8: 2 is lesse then 8, and taken 4 tymes it maketh 8. For the better vnderstandyng of this diffinition, and how this worde Parte, is diuersly taken in Arithmetique and in Geometry, read the declaration of the first diffinition of the 5. booke.

The fourth definition.

4 *Partes are a lesse number in respect of the greater, when the lesse measureth not the greater.*

As the number 3 compared to 5, is partes of 5 and not a part. For the number 3 is lesse then the nūber 5, and doth not measure 5. For taken once it maketh but 3. once 3 is 3, which is lesse then 5. and 3 taken twise maketh 6, which is more then 5. Wherfore it is no part of 5 but partes, namely, three fifth partes of 5. For in the number 3 there are 3 vnities, and euery vnitie is the fifth part of 5. Wherfore 3 is three fifth partes of 5, and so of others.

The fifth definition.

5 *Multiplex is a greater number in comparison of the lesse, when the lesse measureth the greater.*

As 9 compared to 3 is multiplex, the number 9 is greater then the number 3. And moreouer 3 the lesse number measureth 9 the greater number. For 3 taken certaine tymes, namely, 3 tymes maketh 9. three tymes three is 9. For the more ample and full knowledge of this definition, read what is sayd in the explanation of the second definition of the 5 booke, where multiplex is sufficiently entreated of with all his kyndes.

The sixth definition.

6 *An euen number is that, which may be deuided into two equal partes.*

As the number 6 may be deuided into 3 and 3 which are his partes, and they are equall, the one not exceding the other. This definition of Euclide is to be vnderstand of two such equall partes, which ioyned together, make the whole number: as 3 and 3 (the equall partes of 6) ioyned together, make 6, for otherwise many numbers both euen and odde may be deuided into many equall partes, as into 4. 5. 6. or mo, and therfore into 2. As 9 may be deuided into 3 and 3 which are his partes, and are also equall, for the one of them excedeth not the other: yet is not therfore this number 9 an euen number, for that 3 and 3 (these equall partes of 9) added togeather make not 9 but onely 6. Likewise taking the definition so generally, euery number whatsoeuer should be an euen number: for in that sort of vnderstāding there is no number, but that it may be deuided into two equall partes: as this number 7 may be deuided into 3 partes, namely, 3. 1. and 3. of which two, namely, 3 and 3 are equal, yet is not 7 an euen number, because 3 and 3 added together, make not 7. *Boetius* therfore in the first booke of his Arithmetike, for the more playnes, after this maner defineth an euen number: *Numerus par est, qui potest in equalia duo diuidi, sno medio non intercedente*, that is:

Boetius.

An euen number is that which may be deuided into two equall partes, without an vnitie comming betwene them. As 8 is deuided into 4 and 4 two equall partes without an vnitie comming betwene them, which added together, make 8, so that the sence of this definition is, that an euen number is that which is deuided into two such equall partes, which are his two halfe partes.

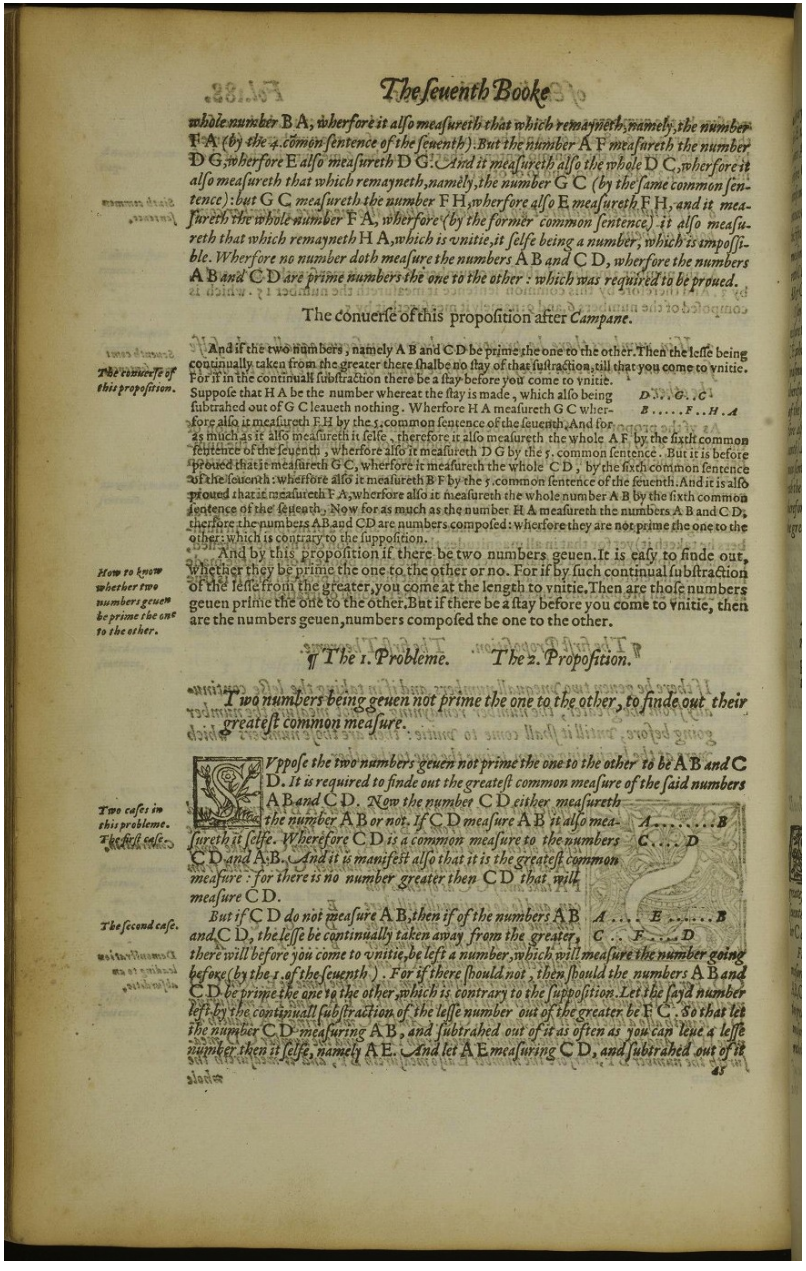
An other definition of an euē number.

Here is also to be noted, that a part is taken in this definition, and in certaine definitions following, not in that signification as it was before defined, namely, for such a part as measureth the whole number, but for any part which helpeth to the making of the whole, and into which the whole may be resolued, so are 3 and 5 partes of 8 in this sence but not in the other sence. For neyther 3 nor 5 measureth 8. Pithagoras and his scholers gaue an other definition of an euen number (which definition *Boetius* also hath) after this maner.

Note.

Pithagoras.

The Euclidean algorithm (Proposition VII.2)



whole number B A, wherfore it alſo meaſureth that which remayneth, namely, the number F A (by the 4. comon ſentence of the ſeuenth). But the number A F meaſureth the number D G, wherfore E alſo meaſureth D G. And it meaſureth alſo the whole D C, wherfore it alſo meaſureth that which remayneth, namely, the number G C (by the ſame common ſentence): but G C meaſureth the number F H, wherfore alſo E meaſureth F H, and it meaſureth the whole number F A, wherfore (by the former common ſentence) it alſo meaſureth that which remayneth H A, which is vnitie, it ſelfe being a number, which is impoſſible. Wherfore no number doth meaſure the numbers A B and C D, wherfore the numbers A B and C D are prime numbers the one to the other: which was required to be proued.

The conuerſe of this propoſition after *Campane*.

The conuerſe of this propoſition.

And if the two numbers, namely A B and C D be prime the one to the other. Then the leſſe being continually taken from the greater there ſhalbe no ſtay of that ſuſtraction, till that you come to vnitie. For if in the continuall ſubſtraction there be a ſtay before you come to vnitie. Suppoſe that H A be the number whereat the ſtay is made, which alſo being ſubtrahed out of G C leaueth nothing. Wherfore H A meaſureth G C wherfore alſo it meaſureth F H by the 5. common ſentence of the ſeuenth. And for as much as it alſo meaſureth it ſelfe, therefore it alſo meaſureth the whole A F by the ſixth common ſentence of the ſeuenth, wherfore alſo it meaſureth D G by the 5. common ſentence. But it is before proued that it meaſureth G C, wherfore it meaſureth the whole C D, by the ſixth common ſentence of the ſeuenth: wherfore alſo it meaſureth B F by the 5. common ſentence of the ſeuenth. And it is alſo proued that it meaſureth F A, wherfore alſo it meaſureth the whole number A B by the ſixth common ſentence of the ſeuenth. Now for as much as the number H A meaſureth the numbers A B and C D, therfore the numbers A B and C D are numbers compoſed: wherfore they are not prime the one to the other: which is contrary to the ſuppoſition.

D . . . G . . C
E . . . . . F . . H . . A

How to know whether two numbers geuen be prime the one to the other.

And by this propoſition if there be two numbers geuen. It is eaſy to finde out, whether they be prime the one to the other or no. For if by ſuch continual ſubſtraction of the leſſe from the greater, you come at the length to vnitie. Then are thoſe numbers geuen prime the one to the other. But if there be a ſtay before you come to vnitie, then are the numbers geuen, numbers compoſed the one to the other.

¶ *The 1. Probleme.* *The 2. Propoſition.*

Two numbers being geuen not prime the one to the other, to finde out their greateſt common meaſure.

Two caſes in this probleme. The firſt caſe.

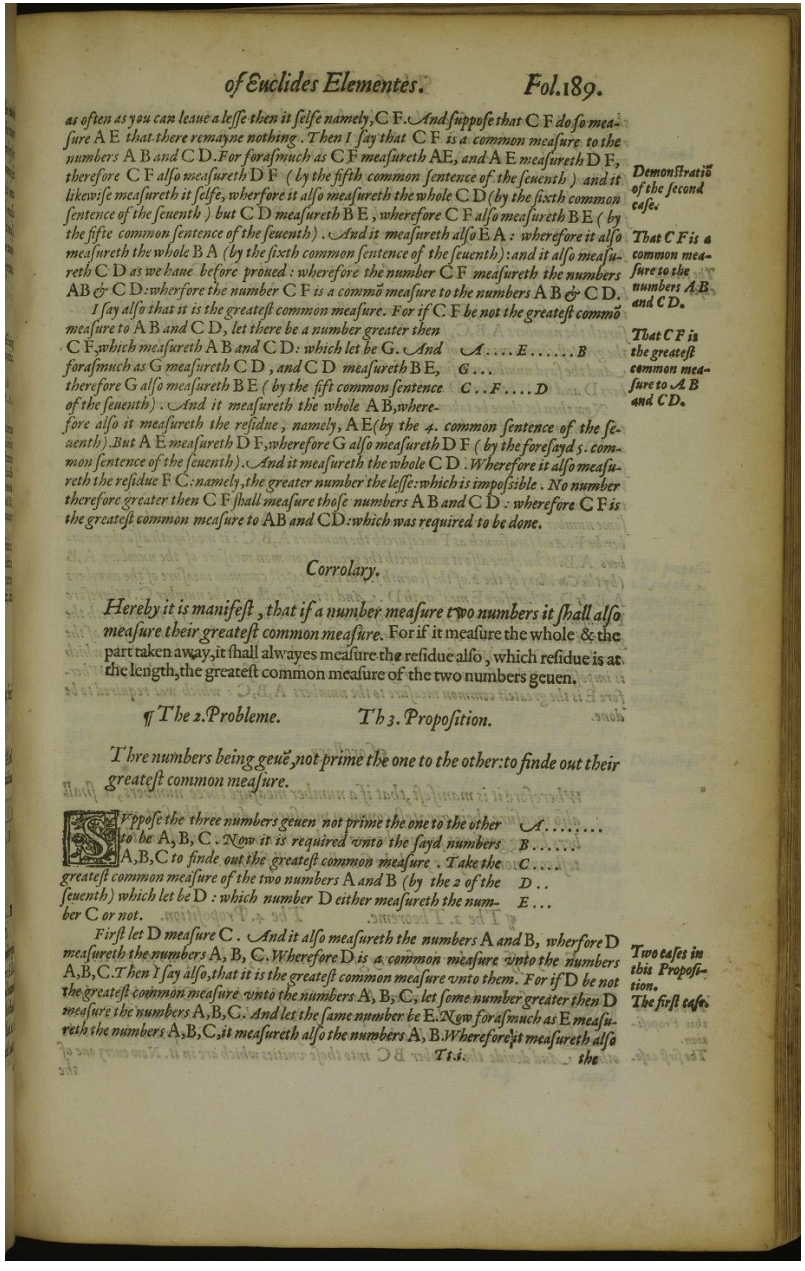
Suppoſe the two numbers geuen not prime the one to the other to be A B and C D. It is required to finde out the greateſt common meaſure of the ſaid numbers A B and C D. Now the number C D either meaſureth the number A B or not. If C D meaſure A B it alſo meaſureth it ſelfe. Wherefore C D is a common meaſure to the numbers C D and A B. And it is manifeſt alſo that it is the greateſt common meaſure: for there is no number greater then C D that will meaſure C D.

A . . . . . B
C . . . D

The ſecond caſe.

But if C D do not meaſure A B, then if of the numbers A B and C D, the leſſe be continually taken away from the greater, there will before you come to vnitie, be left a number, which will meaſure the number going before (by the 1. of the ſeuenth). For if there ſhould not, then ſhould the numbers A B and C D be prime the one to the other, which is contrary to the ſuppoſition. Let the ſayd number left by the continuall ſubſtraction of the leſſe number out of the greater be F C. So that let the number C D meaſuring A B, and ſubtrahed out of it as often as you can leue a leſſe number then it ſelfe, namely A E. And let A E meaſuring C D, and ſubtrahed out of it

A . . . . E . . . . . B
C . . F . . . D



as often as you can leaue a leſſe then it ſelfe namely, C F. And ſuppoſe that C F do ſo meaſure A E that there remayne nothing. Then I ſay that C F is a common meaſure to the numbers A B and C D. For for aſmuch as C F meaſureth A E, and A E meaſureth D F, therefore C F alſo meaſureth D F (by the fifth common ſentence of the ſeuenth) and it likewiſe meaſureth it ſelfe, wherfore it alſo meaſureth the whole C D (by the ſixth common ſentence of the ſeuenth) but C D meaſureth B E, wherefore C F alſo meaſureth B E (by the fifte common ſentence of the ſeuenth). And it meaſureth alſo E A: wherefore it alſo meaſureth the whole B A (by the ſixth common ſentence of the ſeuenth): and it alſo meaſureth C D as we haue before proued: wherefore the number C F meaſureth the numbers A B & C D: wherfore the number C F is a commõ meaſure to the numbers A B & C D.

Demonſtratiõ of the ſecond caſe.

That C F is a common meaſure to the numbers A B and C D.

I ſay alſo that it is the greateſt common meaſure. For if C F be not the greateſt commõ meaſure to A B and C D, let there be a number greater then C F, which meaſureth A B and C D: which let be G. And for aſmuch as G meaſureth C D, and C D meaſureth B E, therefore G alſo meaſureth B E (by the fift common ſentence of the ſeuenth). And it meaſureth the whole A B, wherefore alſo it meaſureth the reſidue, namely, A E (by the 4. common ſentence of the ſeuenth). But A E meaſureth D F, wherefore G alſo meaſureth D F (by the foreſayd 5. common ſentence of the ſeuenth). And it meaſureth the whole C D. Wherefore it alſo meaſureth the reſidue F C: namely, the greater number the leſſe: which is impoſſible. No number therefore greater then C F ſhall meaſure thoſe numbers A B and C D: wherefore C F is the greateſt common meaſure to A B and C D: which was required to be done.

A . . . . E . . . . . B
G . . .
C . . F . . . . D

That C F is the greateſt common meaſure to A B and C D.

Corrolary.

Hereby it is manifeſt, that if a number meaſure two numbers it ſhall alſo meaſure their greateſt common meaſure. For if it meaſure the whole & the part taken away, it ſhall alwayes meaſure the reſidue alſo, which reſidue is at the length, the greateſt common meaſure of the two numbers geuen.

¶ *The 2. Probleme.* *Th 3. Propoſition.*

Thre numbers being geuẽ, not prime the one to the other: to finde out their greateſt common meaſure.

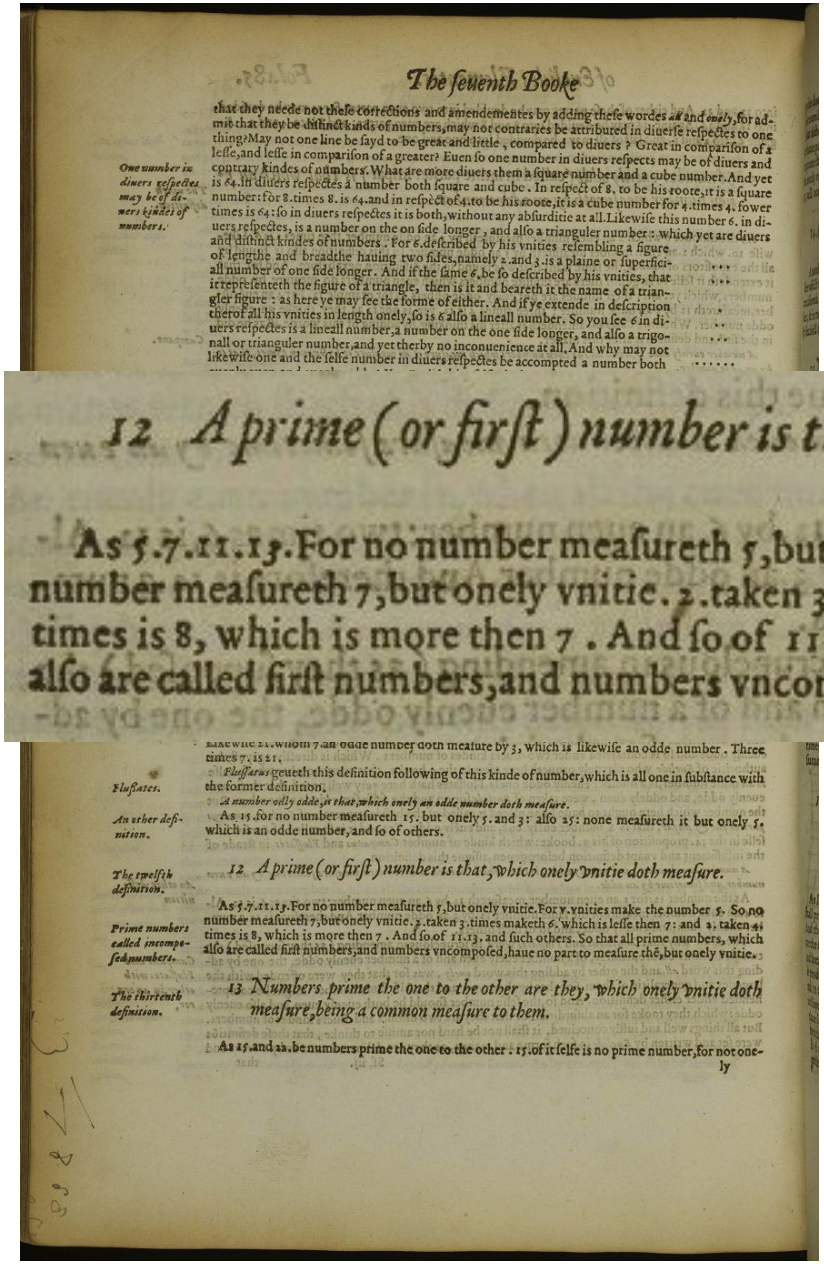
Two caſes in this Propoſition. The firſt caſe.

Suppoſe the three numbers geuen not prime the one to the other to be A, B, C. Now it is required vnto the ſayd numbers A, B, C to finde out the greateſt common meaſure. Take the greateſt common meaſure of the two numbers A and B (by the 2 of the ſeuenth) which let be D: which number D either meaſureth the number C or not.

A . . . . . .
B . . . . . .
C . . . .
D . .
E . . .

First let D meaſure C. And it alſo meaſureth the numbers A and B, wherfore D meaſureth the numbers A, B, C. Wherefore D is a common meaſure vnto the numbers A, B, C. Then I ſay alſo, that it is the greateſt common meaſure vnto them. For if D be not the greateſt common meaſure vnto the numbers A, B, C, let ſome number greater then D meaſure the numbers A, B, C. And let the ſame number be E. Now for aſmuch as E meaſureth the numbers A, B, C, it meaſureth alſo the numbers A, B. Wherefore it meaſureth alſo

Euclid on prime numbers



The ſeuenth Booke

that they neede not theſe corrections and amendementes by adding theſe wordes *all* and *onely*, for admit that they be diſtinct kinds of numbers, may not contraries be attributed in diuerſe reſpectes to one thing? May not one line be ſayd to be great and little, compared to diuers? Great in compariſon of a leſſe, and leſſe in compariſon of a greater? Euen ſo one number in diuers reſpects may be of diuers and contrary kindes of numbers. What are more diuers them a ſquare number and a cube number. And yet is 64. in diuers reſpectes a number both ſquare and cube. In reſpect of 8. to be his roote, it is a ſquare number: for 8. times 8. is 64. and in reſpect of 4. to be his roote, it is a cube number for 4. times 4. tower times is 64: ſo in diuers reſpectes it is both, without any abſurditie at all. Likewiſe this number 6. in diuers reſpectes, is a number on the on ſide longer, and alſo a trianguler number: which yet are diuers and diſtinct kindes of numbers. For 6. deſcribed by his vnities reſembling a figure of lengthe and breadthe hauing two ſides, namely 2. and 3. is a plaine or ſuperficiall number of one ſide longer. And if the ſame 6. be ſo deſcribed by his vnities, that it repreſenteth the figure of a triangle, then is it and beareth it the name of a trianguler figure: as here ye may ſee the forme of either. And if ye extende in deſcription therof all his vnities in length onely, ſo is 6 alſo a lineall number. So you ſee 6 in diuers reſpectes is a lineall number, a number on the one ſide longer, and alſo a trigonall or trianguler number, and yet therby no inconuenience at all. And why may not likewiſe one and the ſelfe number in diuers reſpectes be accompted a number both

One number in diuers reſpectes may be of diuers kindes of numbers.

12 *A prime (or firſt) number is that, which onely vnitie doth meaſure.*

As 5.7.11.13. For no number meaſureth 5, but onely vnitie. For v. vnities make the number 5. So no number meaſureth 7, but onely vnitie. 2. taken 3. times maketh 6. which is leſſe then 7: and 2, taken 4. times is 8, which is more then 7. And ſo of 11.13. and ſuch others. So that all prime numbers, which alſo are called firſt numbers, and numbers vncompoſed, haue no part to meaſure thē, but onely vnitie.

likewiſe 21. whom 7. an odde number doth meaſure by 3, which is likewiſe an odde number. Three times 7. is 21.

Fluſſates. *Fluſſatus* geueth this definition following of this kinde of number, which is all one in ſubſtance with the former definition.

An other definition. *A number odly odde, is that, which onely an odde number doth meaſure.*

As 15. for no number meaſureth 15. but onely 5. and 3: alſo 25: none meaſureth it but onely 5. which is an odde number, and ſo of others.

The twelfth definition. 12 *A prime (or firſt) number is that, which onely vnitie doth meaſure.*

As 5.7.11.13. For no number meaſureth 5, but onely vnitie. For v. vnities make the number 5. So no number meaſureth 7, but onely vnitie. 2. taken 3. times maketh 6. which is leſſe then 7: and 2, taken 4. times is 8, which is more then 7. And ſo of 11.13. and ſuch others. So that all prime numbers, which alſo are called firſt numbers, and numbers vncompoſed, haue no part to meaſure thē, but onely vnitie.

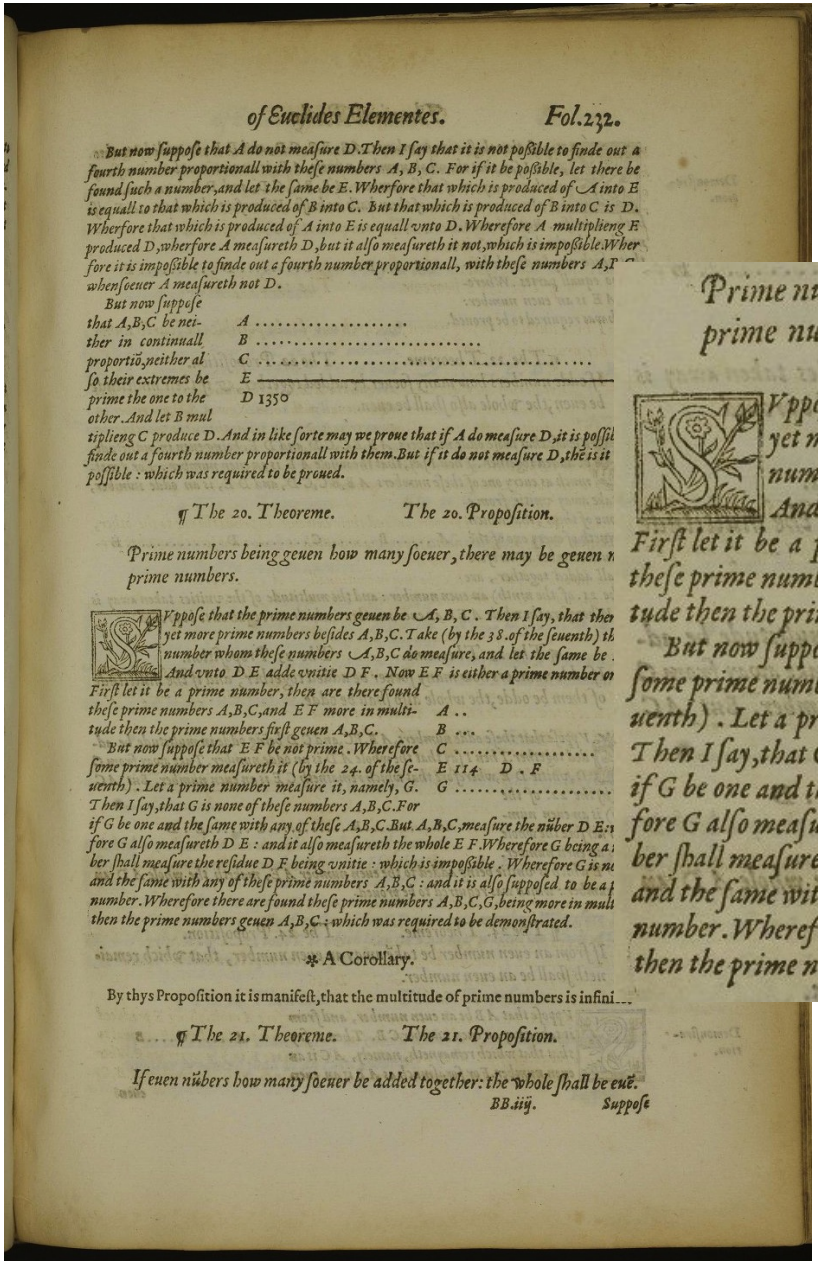
Prime numbers called incompoſed numbers.

The thirteenth definition. 13 *Numbers prime the one to the other are they, which onely vnitie doth meaſure, being a common meaſure to them.*

As 15. and 22. be numbers prime the one to the other. 15. of it ſelfe is no prime number, for not one-

ly

Euclid on prime numbers (Proposition IX.20)



of Euclides Elementes. Fol.232.

But now ſuppoſe that A do not meaſure D. Then I ſay that it is not poſsible to finde out a fourth number proportionall with theſe numbers A, B, C. For if it be poſsible, let there be found ſuch a number, and let the ſame be E. Wherfore that which is produced of A into E is equall to that which is produced of B into C. But that which is produced of B into C is D. Wherfore that which is produced of A into E is equall vnto D. Wherefore A multiplieng E produced D, wherfore A meaſureth D, but it alſo meaſureth it not, which is impoſsible. Wherfore it is impoſsible to finde out a fourth number proportionall, with theſe numbers A, B, C, whenſoeuer A meaſureth not D.

But now ſuppoſe that A, B, C be neither in continuall proportiõ, neither alſo their extremes be prime the one to the other. And let B multiplieng C produce D. And in like ſorte may we proue that if A do meaſure D, it is poſsible to finde out a fourth number proportionall with them. But if it do not meaſure D, thẽ is it poſsible: which was required to be proued.

A ..... B ..... C ..... E ——— D 1350

¶ The 20. Theoreme. The 20. Propoſition.

Prime numbers being geuen how many ſoeuer, there may be geuen more prime numbers.

VPpoſe that the prime numbers geuen be A, B, C. Then I ſay, that there yet more prime numbers beſides A, B, C. Take (by the 38. of the ſeuenth) the number whom theſe numbers A, B, C do meaſure, and let the ſame be D E. And vnto D E adde vnitie D F. Now E F is either a prime number or not. First let it be a prime number, then are there found theſe prime numbers A, B, C, and E F more in multitude then the prime numbers firſt geuen A, B, C.

But now ſuppoſe that E F be not prime. Wherefore ſome prime number meaſureth it (by the 24. of the ſeuenth). Let a prime number meaſure it, namely, G. Then I ſay, that G is none of theſe numbers A, B, C. For if G be one and the ſame with any of theſe A, B, C. But A, B, C, meaſure the nũber D E: wherfore G alſo meaſureth D E: and it alſo meaſureth the whole E F. Wherefore G being a number ſhall meaſure the reſidue D F being vnitie: which is impoſsible. Wherefore G is not one and the ſame with any of theſe prime numbers A, B, C: and it is alſo ſuppoſed to be a prime number. Wherefore there are found theſe prime numbers A, B, C, G, being more in multitude then the prime numbers geuen A, B, C: which was required to be demonſtrated.

A .. B ... C ..... E 114 D . F G .....

✻ A Corollary.

By thys Propoſition it is manifeſt, that the multitude of prime numbers is infini...

¶ The 21. Theoreme. The 21. Propoſition.

If euen nũbers how many ſoeuer be added together: the whole ſhall be euẽ.

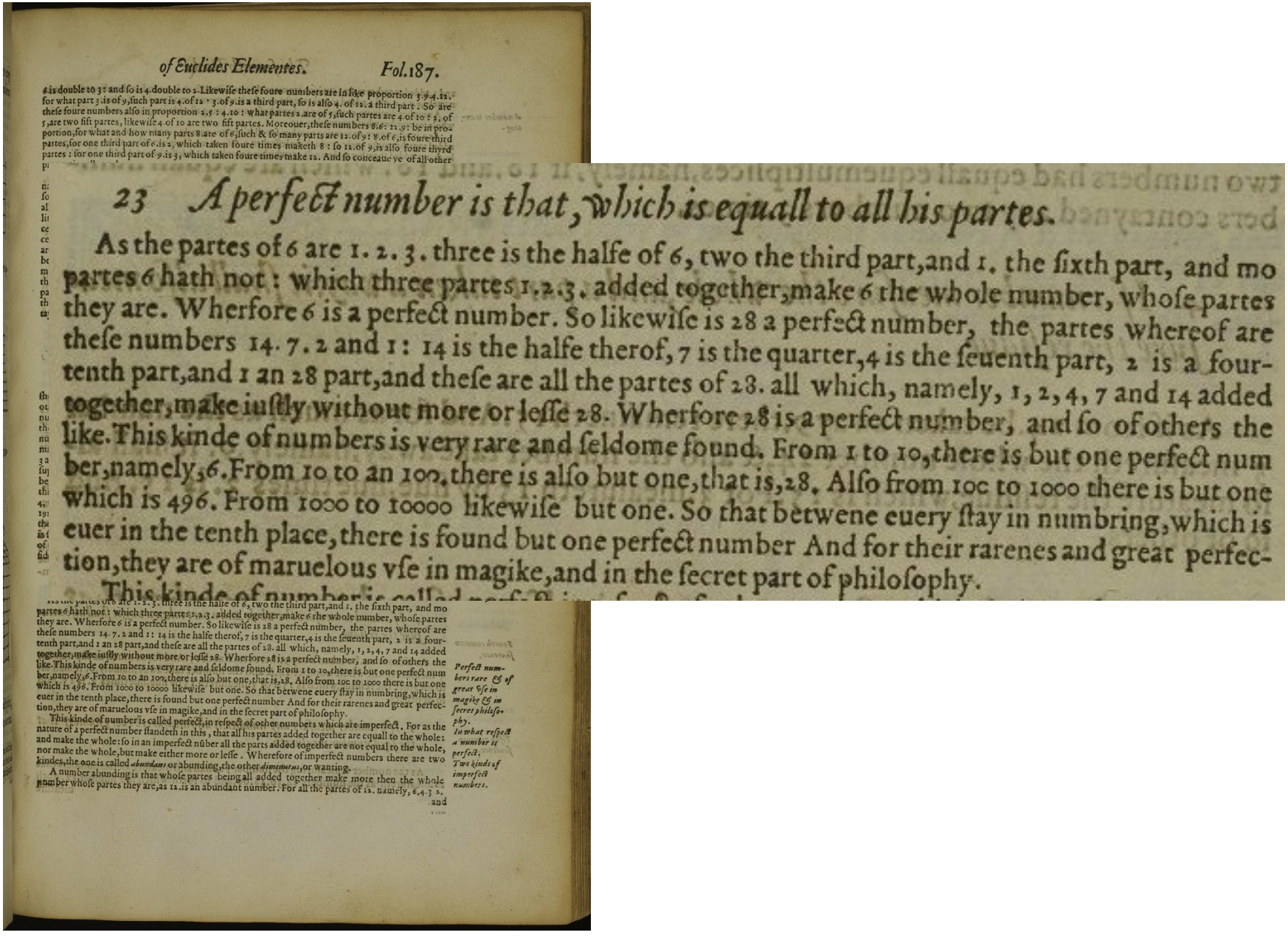
BB.iiij. Suppoſe

Prime numbers being geuen how many ſoeuer, there may be geuen more prime numbers.

VPpoſe that the prime numbers geuen be A, B, C. Then I ſay, that there are yet more prime numbers beſides A, B, C. Take (by the 38. of the ſeuenth) the leſt number whom theſe numbers A, B, C do meaſure, and let the ſame be D E. And vnto D E adde vnitie D F. Now E F is either a prime number or not. First let it be a prime number, then are there found theſe prime numbers A, B, C, and E F more in multitude then the prime numbers firſt geuen A, B, C.

But now ſuppoſe that E F be not prime. Wherefore ſome prime number meaſureth it (by the 24. of the ſeuenth). Let a prime number meaſure it, namely, G. Then I ſay, that G is none of theſe numbers A, B, C. For if G be one and the ſame with any of theſe A, B, C. But A, B, C, meaſure the nũber D E: wherfore G alſo meaſureth D E: and it alſo meaſureth the whole E F. Wherefore G being a number ſhall meaſure the reſidue D F being vnitie: which is impoſsible. Wherefore G is not one and the ſame with any of theſe prime numbers A, B, C: and it is alſo ſuppoſed to be a prime number. Wherefore there are found theſe prime numbers A, B, C, G, being more in multitude then the prime numbers geuen A, B, C: which was required to be demonſtrated.

Euclid on perfect numbers



of Euclides Elementes. Fol. 187.

4 is double to 2: and so is 4 double to 2. Likewise these foure numbers are in like proportion 3.9.4.12. for what part 3 is of 9, such part is 4 of 12 · 3 of 9 is a third part, so is also 4 of 12. a third part. So are these foure numbers also in proportion 2.5 : 4.10 : what partes 2 are of 5, such partes are 4 of 10 : 2. of 5, are two fift partes, likewise 4 of 10 are two fift partes. Moreouer, these numbers 8.6: 12.9: be in proportion, for what and how many parts 8 are of 6, such & so many parts are 12. of 9: 8. of 6, is foure third partes, for one third part of 6. is 2, which taken foure times maketh 8 : so 12. of 9, is also foure thyrd partes : for one third part of 9. is 3, which taken foure times make 12. And so conceaue ye of all other

23 *A perfect number is that, which is equall to all his partes.*

As the partes of 6 are 1. 2. 3. three is the halfe of 6, two the third part, and 1. the sixth part, and mo partes 6 hath not: which three partes 1.2.3. added together, make 6 the whole number, whose partes they are. Wherfore 6 is a perfect number. So likewise is 28 a perfect number, the partes whereof are these numbers 14. 7. 2 and 1: 14 is the halfe therof, 7 is the quarter, 4 is the seuenth part, 2 is a fourtenth part, and 1 an 28 part, and these are all the partes of 28. all which, namely, 1, 2, 4, 7 and 14 added together, make iustly without more or lesse 28. Wherfore 28 is a perfect number, and so of others the like. This kinde of numbers is very rare and seldome found. From 1 to 10, there is but one perfect number, namely, 6. From 10 to an 100, there is also but one, that is, 28. Also from 100 to 1000 there is but one which is 496. From 1000 to 10000 likewise but one. So that betwene euery stay in numbring, which is euer in the tenth place, there is found but one perfect number And for their rarenes and great perfection, they are of maruelous vse in magike, and in the secret part of philosophy.

Perfect numbers rare & of great vse in magike & in secret philosophy.

This kinde of number is called perfect, in respect of other numbers which are imperfect. For as the nature of a perfect number standeth in this, that all his partes added together are equall to the whole: and make the whole: so in an imperfect nũber all the parts added together are not equal to the whole, nor make the whole, but make either more or lesse. Wherefore of imperfect numbers there are two kindes, the one is called *abundans* or abunding, the other *diminutus*, or wanting.

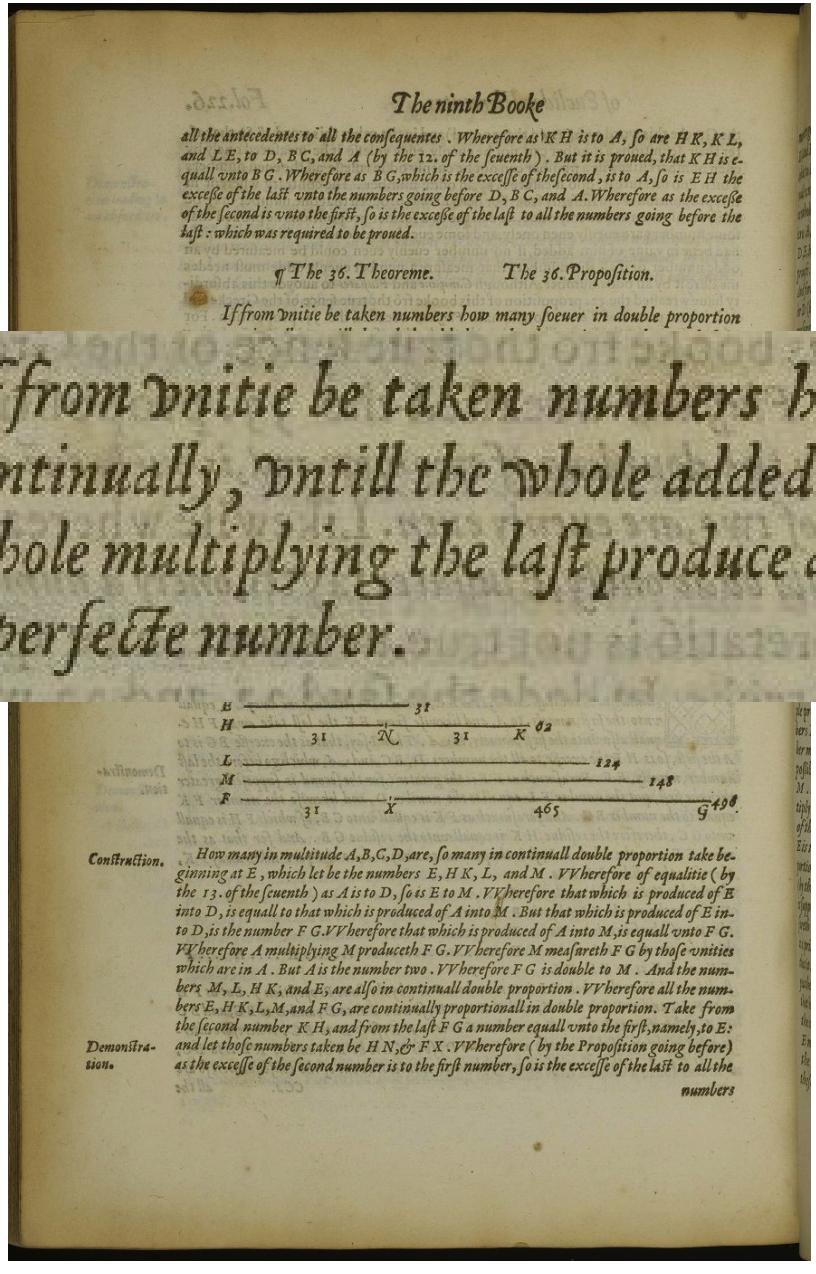
In what respect a number is perfect.

A number abunding is that whose partes being all added together make more then the whole number whose partes they are, as 12. is an abundant number. For all the partes of 12. namely, 6.4.3 2.

Two kinds of imperfect numbers.

and

Euclid on perfect numbers (Proposition IX.36)



The ninth Booke

all the antecedentes to all the conſequentes. Wherefore as K H is to A, ſo are H K, K L, and L E, to D, B C, and A (by the 12. of the ſeuenth). But it is proued, that K H is equall vnto B G. Wherefore as B G, which is the exceſſe of the ſecond, is to A, ſo is E H the exceſſe of the laſt vnto the numbers going before D, B C, and A. Wherefore as the exceſſe of the ſecond is vnto the firſt, ſo is the exceſſe of the laſt to all the numbers going before the laſt: which was required to be proued.

¶ The 36. Theoreme. The 36. Propoſition.

If from vnitie be taken numbers how many ſoeuer in double proportion

If from vnitie be taken numbers how many ſoeuer in double proportion continually, vntill the whole added together be a prime number, and if the whole multiplying the laſt produce any number, that which is produced is a perfecte number.

E 31
H 31 N 31 K 62
L 124
M 148
F 31 X 465 G 496

Conſtruction. How many in multitude A, B, C, D, are, ſo many in continuall double proportion take beginning at E, which let be the numbers E, H K, L, and M. VVherefore of equalitie (by the 13. of the ſeuenth) as A is to D, ſo is E to M. VVherefore that which is produced of E into D, is equall to that which is produced of A into M. But that which is produced of E into D, is the number F G. VVherefore that which is produced of A into M, is equall vnto F G. VVherefore A multiplying M produceth F G. VVherefore M meaſureth F G by thoſe vnities which are in A. But A is the number two. VVherefore F G is double to M. And the numbers M, L, H K, and E, are alſo in continuall double proportion. VVherefore all the numbers E, H K, L, M, and F G, are continually proportionall in double proportion. Take from the ſecond number K H, and from the laſt F G a number equall vnto the firſt, namely, to E: and let thoſe numbers taken be H N, & F X. VVherefore (by the Propoſition going before) as the exceſſe of the ſecond number is to the firſt number, ſo is the exceſſe of the laſt to all the

Demonſtration.

numbers

In modern terms: if $2^n - 1$ is prime, then $2^{n-1}(2^n - 1)$ is perfect

Number theory after Euclid

Very little for many centuries...

Recall that Diophantus' *Arithmetica* (13 books, c. AD 250) featured number problems; for example [from Lecture IX]:

> Problem I.27: *Find two numbers such that their sum and product are given numbers*

The *Arithmetica* also features problems and ideas that we would now classify as number-theoretic; for example:

> Problem III.19: *To find four numbers such that the square of their sum plus or minus any one singly gives a square*
>
> Problem V.9: *To divide unity into two parts such that, if a given number is added to either part, the result will be a square*

Restrictions on the permitted form of solutions to problems eventually gave rise to the notion of Diophantine equations

Number theory outside Europe

Sūnzǐ Suànjīng 孙子算经 (*The Mathematical Classic of Master Sun*) (3rd–5th century BC) contains a statement, but no proof, of the Chinese Remainder Theorem for the solution of simultaneous congruences

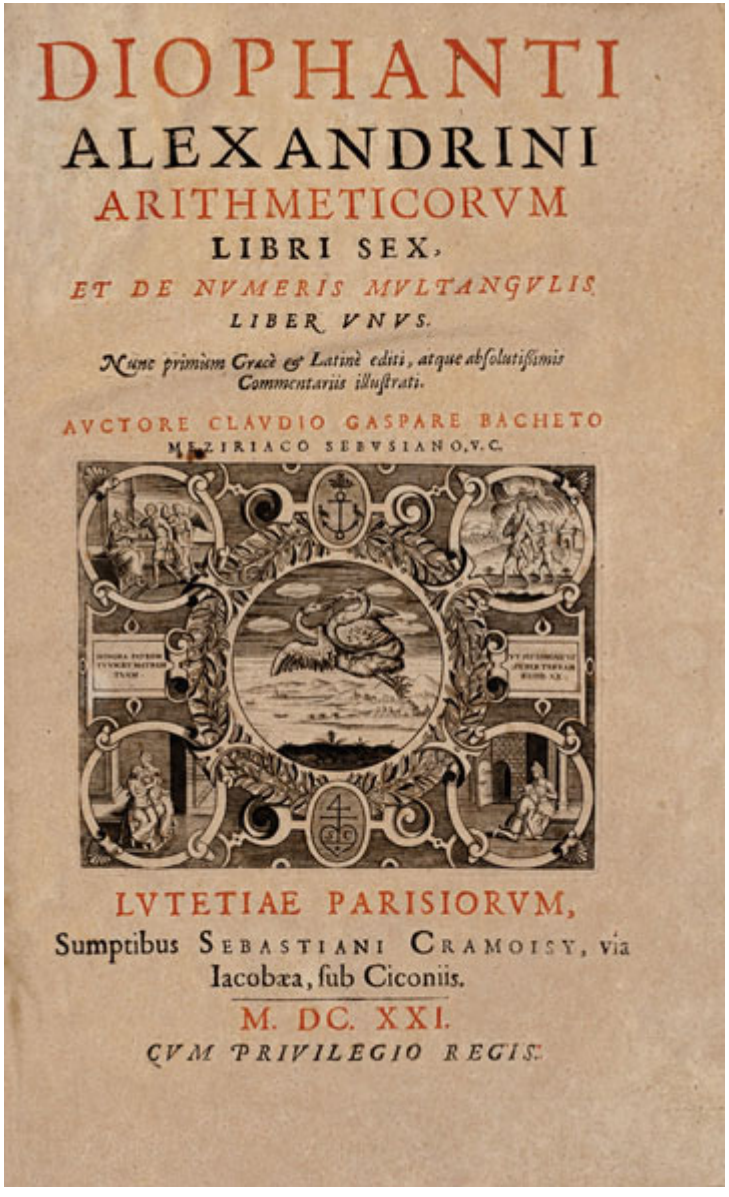
An algorithm for the solution was provided by Aryabhata in 6th-century India

In 7th-century India, Brahmagupta studied Diophantine equations (including Pell's equation — see later, and also: Toke Knudsen and Keith Jones, 'The Pell Equation in India', 2017)

These works were unknown in Europe until the 19th century

See: Eva Caianiello, 'Indeterminate linear problems from Asia to Europe', *Lettera Matematica* 6 (2018), 233–243

17th-century number theory


DIOPHANTI
ALEXANDRINI
ARITHMETICORVM
LIBRI SEX.
ET DE NVMERIS MVLTANGVLIS
LIBER VNVS.
Nunc primùm Græcè & Latinè editi, atque absolutissimis Commentariis illustrati.
AVCTORE CLAVDIO GASPARE BACHETO
MEZIRIACO SEBVSIANO, V.C.
LVTETIAE PARISIORVM,
Sumptibus SEBASTIANI CRAMOISY, via Iacobæa, sub Ciconiis.
M. DC. XXI.
CVM PRIVILEGIO REGIS.

Bachet's Latin edition of Diophantus' *Arithmetica* (1621)

Pierre de Fermat owned a 1637 edition, which he studied and annotated

Fermat on number theory

Fermat's Little Theorem: if a is any integer and p is prime then p divides $a^p - a$

Studies of 'Pell's Equation' $x^2 - Dy^2 = 1$

Conjectures on perfect numbers [more in a moment]

Studies of diophantine problems leading to 'Fermat's Last Theorem' [more in a moment]

Published nothing — had to be exhorted to write his ideas down

(See *Mathematics emerging*, §§6.1–6.3)

The 'Last Theorem'

Arithmetica Problem II.8 concerns the splitting of a given square number into two other squares

Fermat's marginal note:

> *It is impossible to separate a cube into two cubes, or a fourth power into two fourth powers, or in general, any power higher than the second, into two like powers. I have discovered a truly marvelous proof of this, which this margin is too narrow to contain.*

(See: Simon Singh, *Fermat's Last Theorem*, Fourth Estate, 1998)

Perfect numbers

Euclid's Theorem: if $2^n - 1$ is prime then $2^{n-1}(2^n - 1)$ is perfect

Fermat to Mersenne (1640): if $2^n - 1$ is prime then n must be prime

Mersenne (1644): if $p \leq 257$ and $2^p - 1$ is prime then p is one of 2, 3, 5, 7, 13, 17, 67 (a misprint for 61 perhaps?), 127, 257. Not quite right: $2^{89} - 1$, $2^{107} - 1$ are prime and $2^{257} - 1$ is composite.

Euler: proof that all even perfect numbers are of Euclid's form (proved 1749, but published posthumously)

(See *Mathematics emerging*, §6.1.2)

NB. 51 Mersenne primes are currently known, the largest being $2^{82,589,933} - 1$ (found in June 2019)

17th-century attitudes to number theory

Fermat failed to spark an interest in number theory in his contemporaries

Pascal to Fermat (1655):

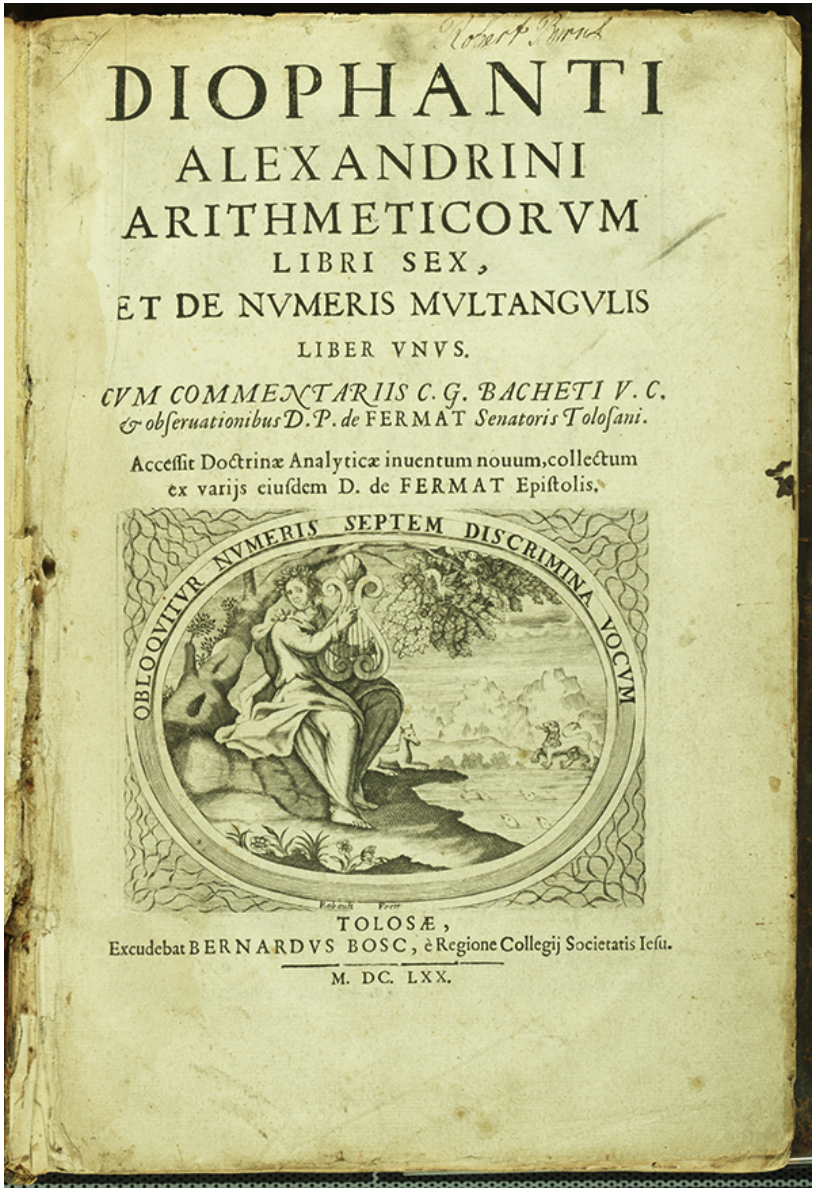
> *. . . seek elsewhere those who can follow you in your numerical discoveries . . . I confess to you that this goes far beyond me . . .*

Number-theoretic investigations were widely regarded as trivial and uninteresting

Huygens to Wallis:

> *There is no lack of better topics for us to spend our time on . . .*

The ‘rebirth’ of number theory


DIOPHANTI
ALEXANDRINI
ARITHMETICORVM
LIBRI SEX,
ET DE NVMERIS MVLTANGVLIS
LIBER VNVS.
CVM COMMENTARIIS C. G. BACHETI V. C.
& obſeruationibus D. P. de FERMAT Senatoris Toloſani.
Acceſſit Doctrinæ Analyticæ inuentum nouum, collectum
ex varijs eiuſdem D. de FERMAT Epiſtolis.
OBLOQVITVR NVMERIS SEPTEM DISCRIMINA VOCVM
TOLOSÆ,
Excudebat BERNARDVS BOSC, è Regione Collegij Societatis Ieſu.
M. DC. LXX.

1670 edition of Bachet, published by Samuel Fermat, including his father’s notes

The ‘Last Theorem’ was not the only result for which Fermat failed to provide a proof

Number theory was ‘reborn’ from the attempts of Euler (and later Lagrange and Legendre) to fill the gaps left by Fermat

Euler on number theory

Euler (1747):

> *Nor is the author disturbed by the authority of the greatest mathematicians when they sometimes pronounce that number theory is altogether useless and does not deserve investigation. In the first place, knowledge is always good in itself, even when it seems to be far removed from common use. Secondly, all the aspects of the truth which are accessible to our mind are so closely related to one another that we dare not reject any of them as being altogether useless. . . .*
>
> *Consequently, the present author considers that he has by no means wasted his time and effort in attempting to prove various theorems concerning integers and their divisors. . . . Moreover, there is little doubt that the method used here by the author will turn out to be of no small value in other investigations of greater import.*

19th-century number theory

Gauss's *Disquisitiones arithmeticae* (1801) became a key text for many years to come: modular arithmetic, quadratic forms, cyclotomy, ...

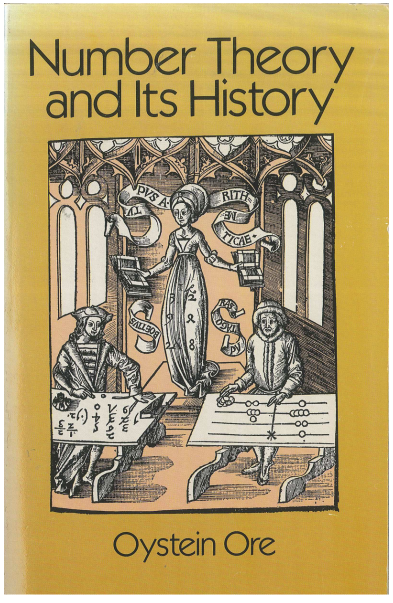
Number-theoretic problems (especially attempts to prove Fermat's Last Theorem) led to the development of ideal theory, and the linking of number theory and abstract algebra in algebraic number theory

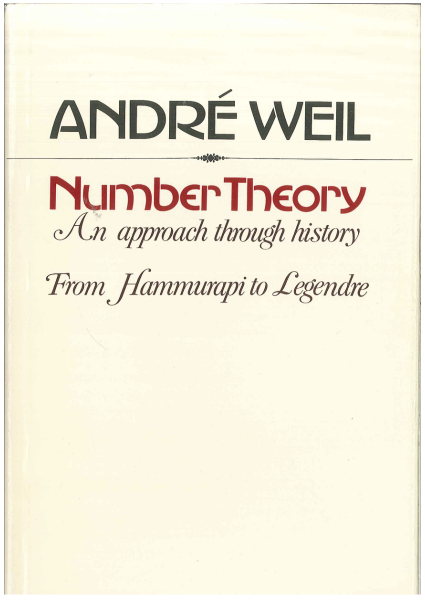
By the end of the 19th century, a new branch, analytic number theory, had also emerged (e.g., Riemann hypothesis, Prime Number Theory $\pi(x) \sim \frac{x}{\log x}$, ...)

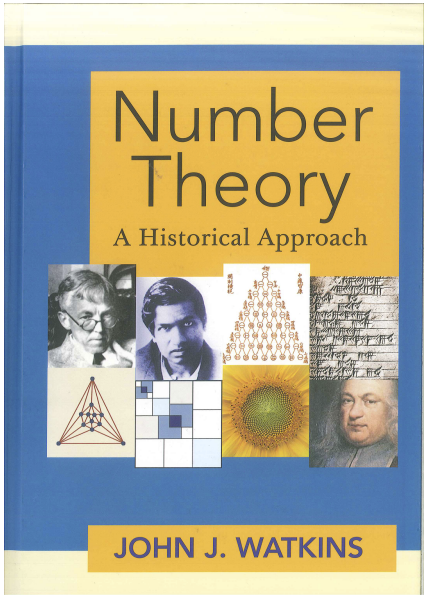
Sophie Germain (1776–1831)

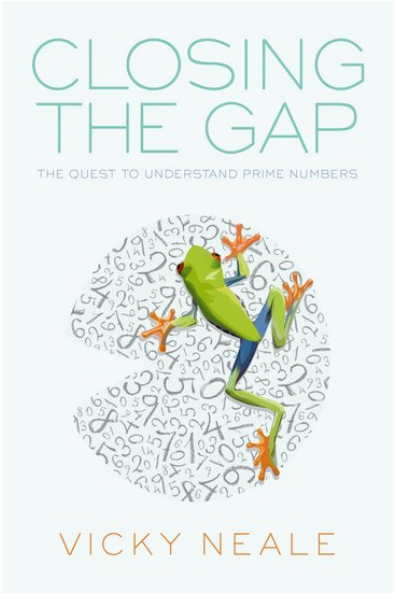
- Largely self-taught in mathematics by reading the works of Newton, Euler, and others
- Corresponded with Lagrange, Legendre, Gauss
- Pioneer of elasticity theory, but also solved some cases of Fermat's Last Theorem
- See: Jenny Boucard, 'Arithmetic and memorial practices by and around Sophie Germain in the 19th century', in *Against all odds: Women's ways to mathematical research since 1800* (ed. Eva Kaufholz-Soldat & Nicola M. R. Ostwald), Springer, 2020, pp. 185–230

The history of number theory









Leonard Eugene Dickson, *History of the theory of numbers*, 3 vols., Carnegie Institution of Washington, 1919–1923: I, II, III

BO1 History of Mathematics
Lecture XVI
Concluding miscellany

MT 2022 Week 8

Summary

- The exam (briefly)
- Points to ponder
- The history of the history of mathematics*
- Hilary Term reading course

Structure of the exam paper

Section A

- Six extracts given
- Choose two and comment on the context, content, and significance
- Each extract is worth 25 marks
- Each extract is typically one short paragraph — it will relate to a topic that we have studied, though you may not have seen the precise extract before
- By way of practice, choose any quotation or short extract that has appeared on the lecture slides

Section B

- Three essay topics given
- Choose one
- Answer worth 50 marks

Typical exam questions (Section B)

Q. Discuss, with reference to specific examples, how concept X (or terminology Y, or notation Z, ...) has developed between 1600 and 1900.

Q. Discuss with reference to specific examples, how attitudes towards X have changed between 1600 and 1900.

Q. Discuss the significance of text X.

Q. Describe some aspects of the work of major figure X.

Points to ponder (1)

What is the history of mathematics?

What does it mean to study the history of mathematics?

What is mathematics?

Points to ponder (2)

What do you think the words ‘mathematics’ and ‘mathematician’ have meant throughout this course?

Have they had the same meanings throughout?

More generally, have they had the same meanings throughout history?

Points to ponder (3)

If we choose to understand the word ‘mathematics’ differently, how does this change our view of the history of mathematics?

How could a revised definition of ‘mathematics’ change the selection of people and cultures who appear in the story?

What does the study of the history of mathematics have to tell us about the way in which we approach mathematics nowadays?

Historiography of mathematics

According to the *OED*:

historiography, *n.*

1. The writing of history; written history.
2. The study of history-writing, esp. as an academic discipline.

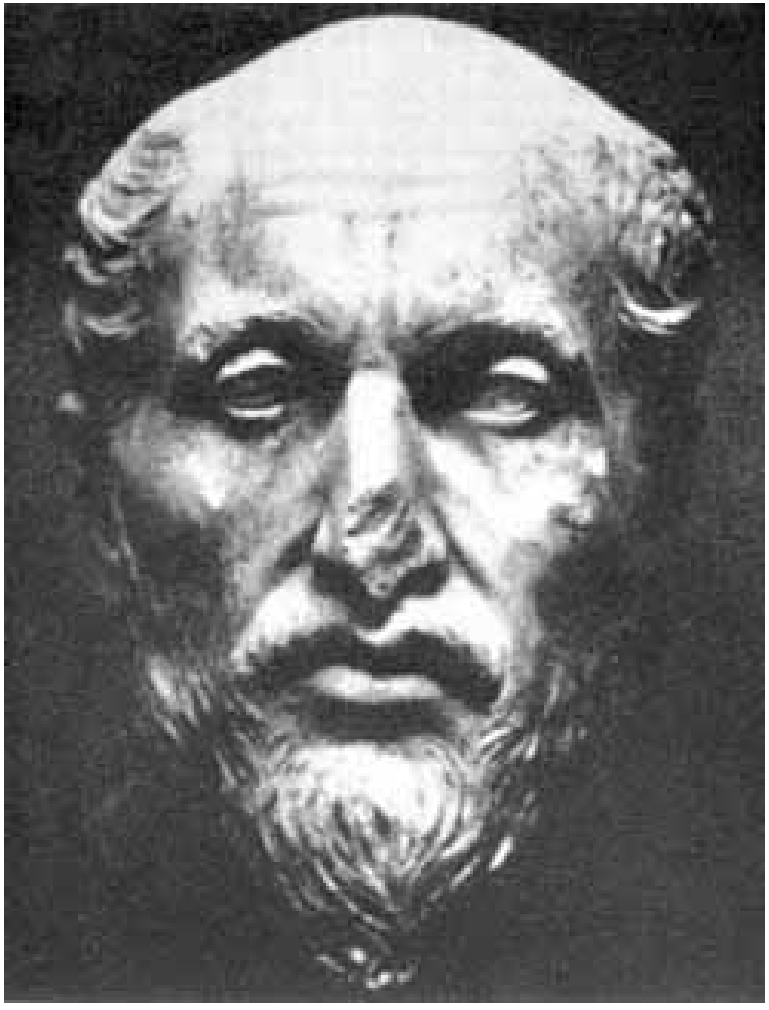
Ancient histories of mathematics

Aristotle (384–322 BC)

Eudemus (4th century BC)

- Student and editor of Aristotle
- *History of Arithmetic*
- *History of Geometry*
- *History of Astronomy*

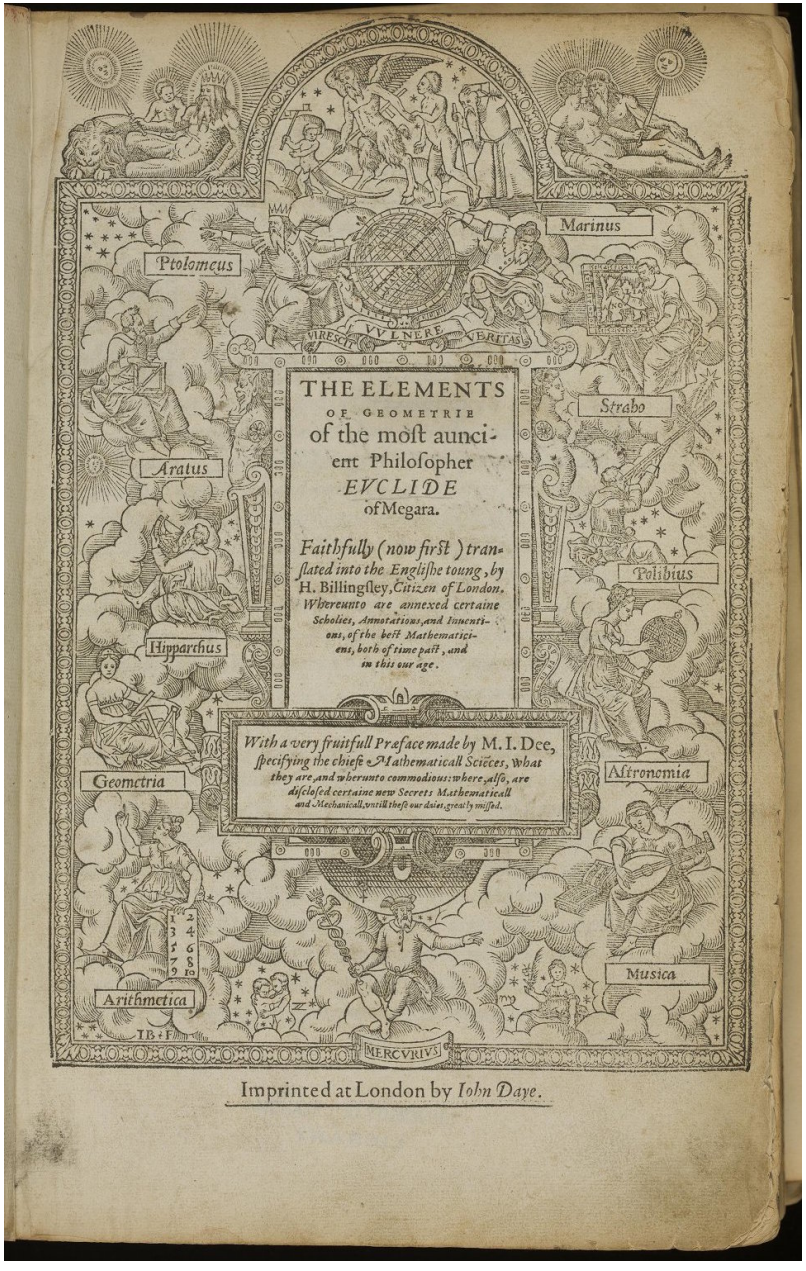
Biographical background



Proclus's commentary on Euclid's *Elements* (5th century AD)

- (Spurious?) biographical details
- Built on anecdotes provided by Pappus (4th century AD)

Later historical attributions


Ptolomeus
Marinus
VIRESCIT VVLNERE VERITAS
THE ELEMENTS
OF GEOMETRIE
of the most auncient Philosopher
EVCLIDE
of Megara.
Faithfully (now first) translated into the Englishe toung, by H. Billingsley, Citizen of London.
Whereunto are annexed certaine Scholies, Annotations, and Inventions, of the best Mathematicians, both of time past, and in this our age.
Strabo
Aratus
Polibius
Hipparchus
With a very fruitfull Preface made by M. I. Dee, specifying the chiefe Mathematicall Sciences, what they are, and wherunto commodious: where, also, are disclosed certaine new Secrets Mathematicall and Mechanicall, until these our dayes greatly missed.
Geometria
Astronomia
Musica
Arithmetica
MERCVRIVS
Imprinted at London by Iohn Daye.

a full understanding of geometry
"requireth diligent studie and reading of olde auncient authors"

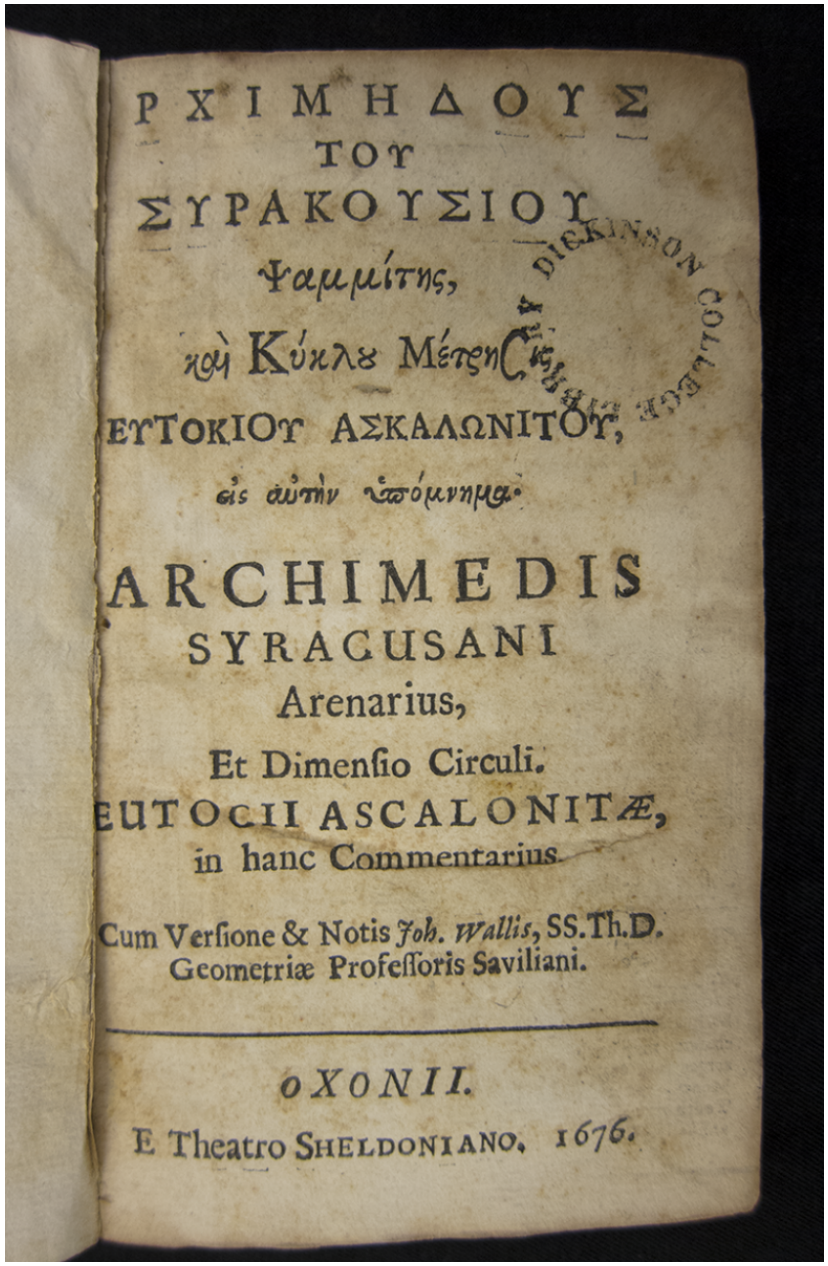
Renaissance humanist attitudes

Sir Henry Savile (1549–1622)

The teaching of mathematics should be founded on humanist principles:

- it should stem from the works of classical antiquity;
- scholars ought to have a concern for the history of their subject;
- they should actively seek to restore and edit surviving texts.

Renaissance humanist attitudes



ΡΧΙΜΗΔΟΥΣ
ΤΟΥ
ΣΥΡΑΚΟΥΣΙΟΥ
Ψαμμίτης,
καὶ Κύκλου Μέτρησις.
ΕΥΤΟΚΙΟΥ ΑΣΚΑΛΩΝΙΤΟΥ,
εἰς αὐτὴν ὑπόμνημα.

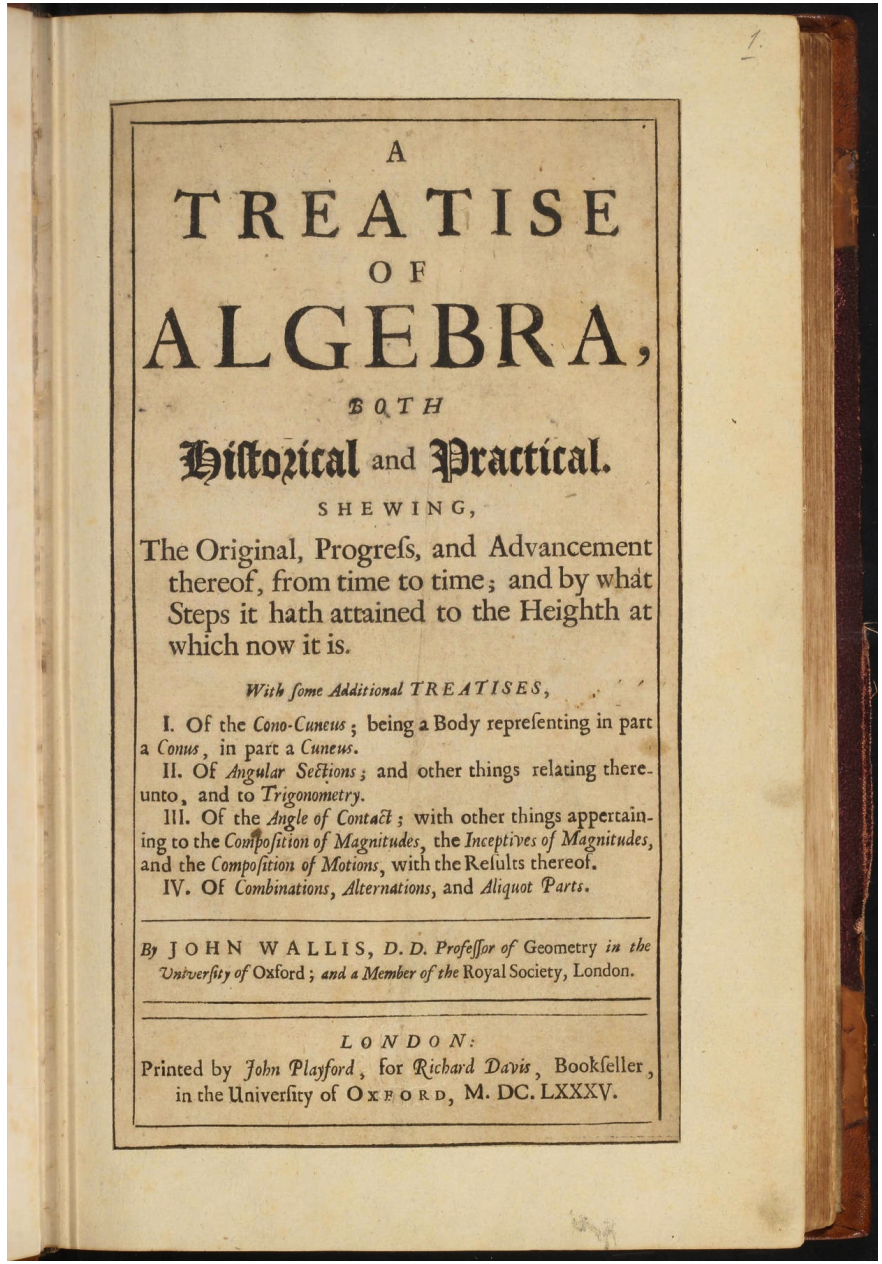
ARCHIMEDIS
SYRACUSANI
Arenarius,
Et Dimensio Circuli.
EUTOCII ASCALONITÆ,
in hanc Commentarius.

Cum Versione & Notis *Joh. Wallis*, SS.Th.D.
Geometriæ Professoris Saviliani.

OXONII.
E Theatro SHELDONIANO. 1676.

DICKINSON COLLEGE LIBRARY

Nationalist attitudes



A

TREATISE

OF

ALGEBRA,

BOTH

Hiſtorical and Practical.

SHEWING,

The Original, Progreſs, and Advancement thereof, from time to time; and by what Steps it hath attained to the Heighth at which now it is.

With ſome Additional TREATISES,

I. Of the *Cono-Cuneus*; being a Body repreſenting in part a *Conus*, in part a *Cuneus*.

II. Of *Angular Sections*; and other things relating thereunto, and to *Trigonometry*.

III. Of the *Angle of Contact*; with other things appertaining to the *Compoſition of Magnitudes*, the *Inceptives of Magnitudes*, and the *Compoſition of Motions*, with the Reſults thereof.

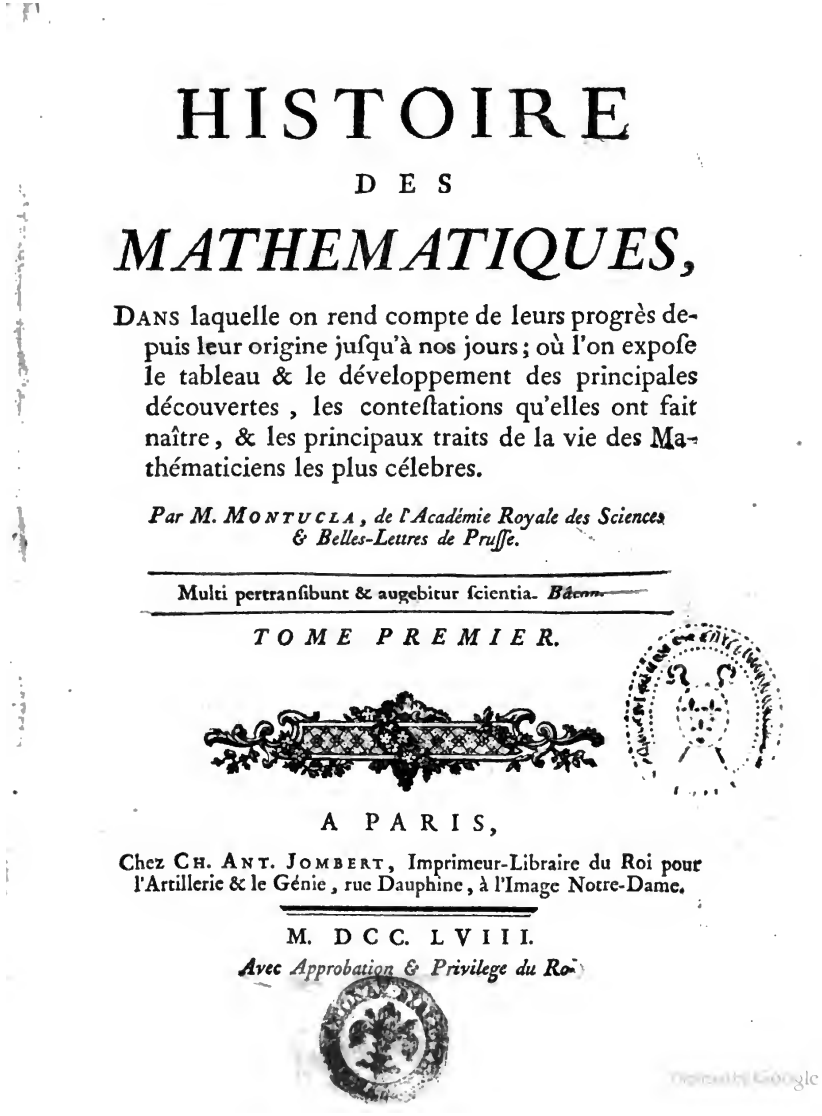
IV. Of *Combinations*, *Alternations*, and *Aliquot Parts*.

By JOHN WALLIS, *D. D. Profeſſor of* Geometry *in the Univerſity of* Oxford; *and a Member of the* Royal Society, London.

LONDON:

Printed by *John Playford*, for *Richard Davis*, Bookſeller, in the Univerſity of OXFORD, M. DC. LXXXV.

Comprehensive histories of mathematics



HISTOIRE

DES

MATHEMATIQUES,

DANS laquelle on rend compte de leurs progrès depuis leur origine juſqu'à nos jours; où l'on expoſe le tableau & le développement des principales découvertes, les conteſtations qu'elles ont fait naître, & les principaux traits de la vie des Mathématiciens les plus célebres.

Par M. MONTUCLA, de l'Académie Royale des Sciences & Belles-Lettres de Pruſſe.

Multi pertranſibunt & augebitur ſcientia. *Bacon.*

TOME PREMIER.

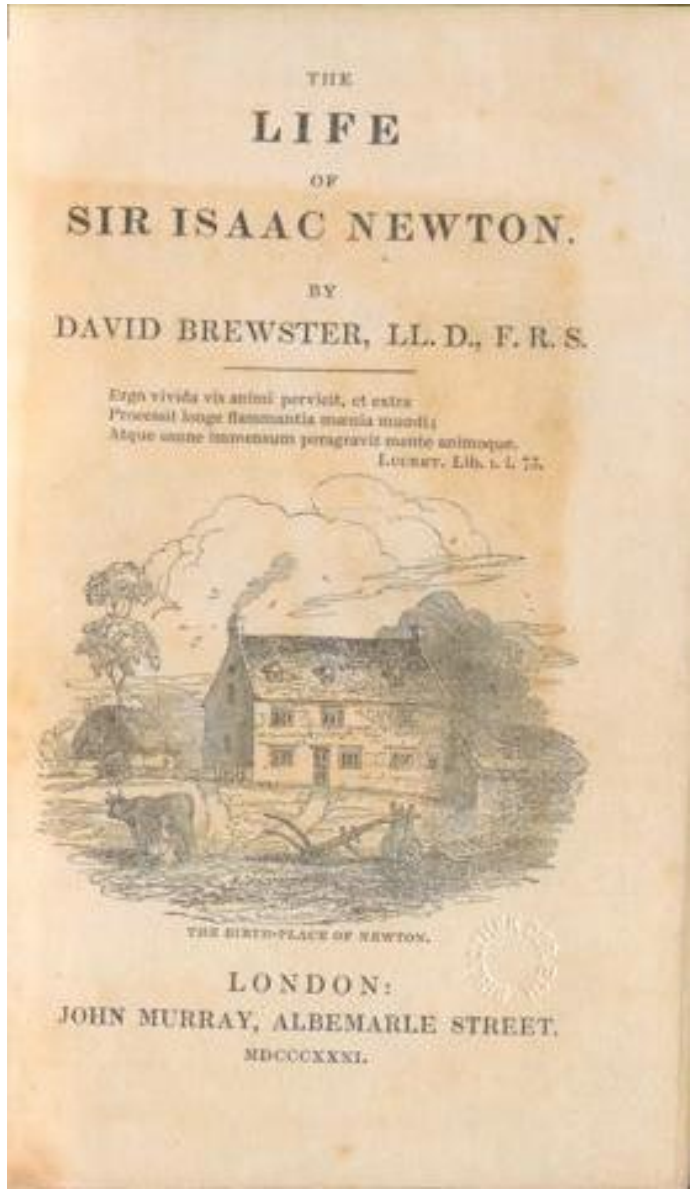
A PARIS,

Chez CH. ANT. JOMBERT, Imprimeur-Libraire du Roi pour l'Artillerie & le Génie, rue Dauphine, à l'Image Notre-Dame.

M. DCC. LVIII.

Avec Approbation & Privilege du Ro-

Lauding the great mathematicians



THE

LIFE

OF

SIR ISAAC NEWTON.

BY

DAVID BREWSTER, LL.D., F.R.S.

Ergò vivida vis animi pervicit, et extra
Processit longè flammantia mœnia mundi;
Atque omne immensum peragravit mente animoque.
LUCRET. Lib. i. l. 73.

THE BIRTH-PLACE OF NEWTON.

LONDON:

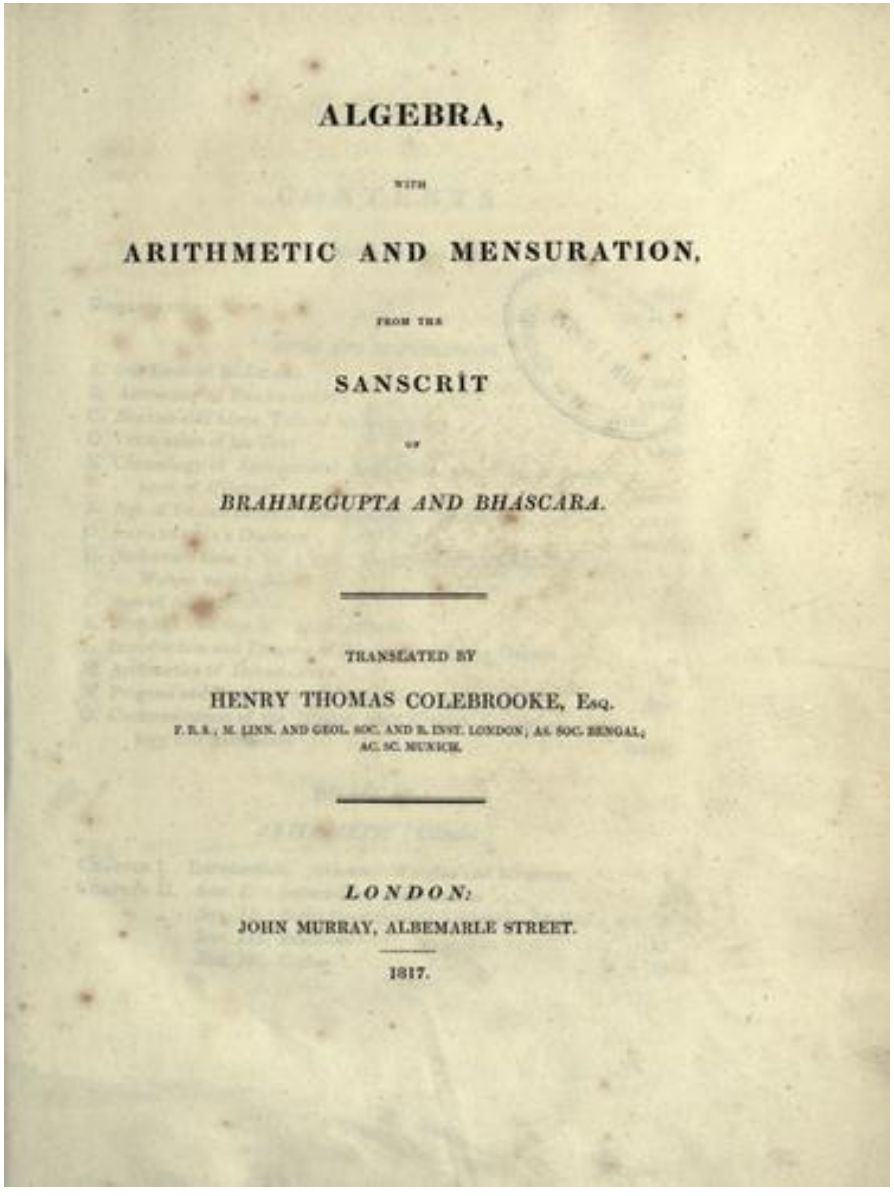
JOHN MURRAY, ALBEMARLE STREET.

MDCCCXXXI.

Adding greater nuance

See: Adrian Rice, 'Augustus De Morgan: historian of science', *History of Science* 34 (1996), 201–240

Awareness of mathematics beyond Europe



ALGEBRA,

WITH

ARITHMETIC AND MENSURATION,

FROM THE

SANSCRIT

OF

BRAHMEGUPTA AND BHASCARA.

TRANSLATED BY

HENRY THOMAS COLEBROOKE, Esq.

F. R. S.; M. LINN. AND GEOL. SOC. AND R. INST. LONDON; AS. SOC. BENGAL; AC. SC. MUNICH.

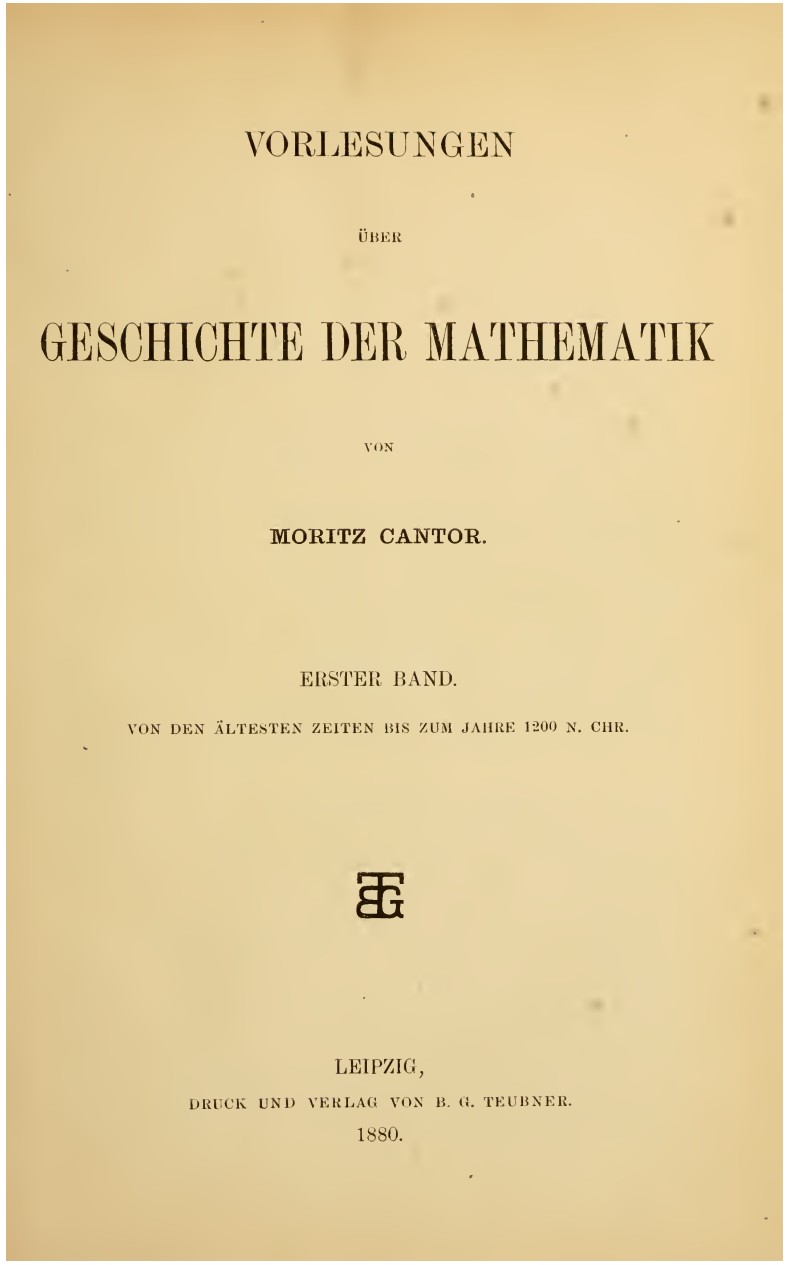
LONDON:

JOHN MURRAY, ALBEMARLE STREET.

1817.

See: Ivahn Smadja, 'Sanskrit versus Greek 'proofs' : history of mathematics at the crossroads of philology and mathematics in nineteenth-century Germany', *Revue d'histoire des mathématiques* 21(2) (2015) 217–349

Anecdotal history



VORLESUNGEN

ÜBER

GESCHICHTE DER MATHEMATIK

VON

MORITZ CANTOR.

ERSTER BAND.

VON DEN ÄLTESTEN ZEITEN BIS ZUM JAHRE 1200 N. CHR.

LEIPZIG,

DRUCK UND VERLAG VON B. G. TEUBNER.

1880.

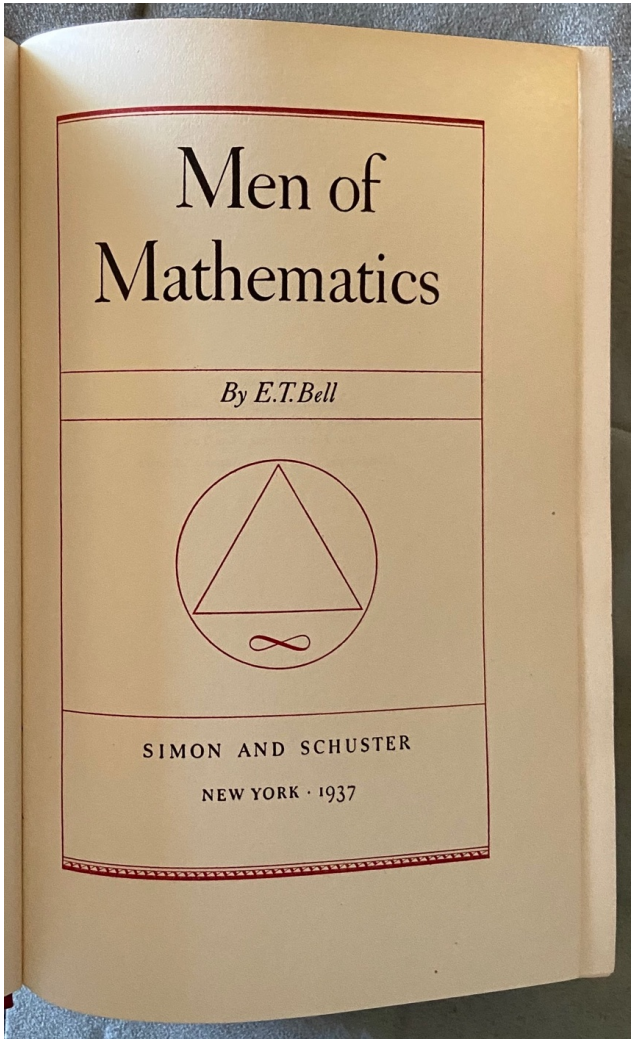
Who studies the history of mathematics?

- Mathematicians?
  - Because mathematical knowledge is key?
  - Because only mathematicians will read it?
  - Suitable retirement project?
- Historians?
  - Because only an understanding of historical context makes it history?
  - For better integration into wider historical scholarship?
- Scholars who are somewhere in between?

Professionalisation

Otto Neugebauer (1899–1990)

Popularisation



- Humanisation of mathematics
- Mathematical myth-making
- Mathematical anecdotes for community-building
- Drawback: reinforcement of a particular view of the subject

Rewriting the history of mathematics

until you obtain 2. It was therefore indeed a new idea to duplicate $\bar{n}$ by dividing 2 by n.

We do not know in whose brain this thought arose for the first time, nor when this happened. It certainly occurred long before the era of our texts, for the $(2 : n)$-table of the Rhind papyrus, which includes all the odd numbers from $n = 3$ to $n = 101$, was not constructed all at one time; its separate parts were computed by different methods. The oldest section contains the denominators which are divisible by 3; without exception, they all proceed according to the same rule:

$$2 : 9 = \bar{6} + \overline{18}$$
$$2 : 15 = \overline{10} + \overline{30}$$
$$2 : 21 = \overline{14} + \overline{42}.$$

In these cases the division $2 : 3k$ is simply a confirmation of a known result. In the other cases (certainly from $n = 11$ on), the duplication appears to have been obtained by actually carrying out the division of 2 by n. The text exhibits the divisions more or less explicitly, as in the following examples $(2 : 5$ and $2 : 7)$.

What part is 2 of 5? $\bar{3}$ is $1 + \bar{3}$, $\overline{15}$ is $\bar{3}$,

i.e. a third of 5 is $1 + \bar{3}$, a fifteenth is $\bar{3}$; these add up to 2. The result of the division is therefore $\bar{3} + \overline{15}$; the terms $\bar{3}$ and $\overline{15}$ are clearly visible because they are written in red. In our "translation" the red symbols have been printed in bold-face type.

Computation:

1	5
$\bar{3}$	$3 + \bar{3}$
/ $\bar{3}$	$1 + \bar{3}$
/ $\overline{15}$	$\bar{3}$

What part is 2 of 7? $\bar{4}$ is $1 + \bar{2} + \bar{4}$, $\overline{28}$ is $\bar{4}$.

Computation:

1	7		
$\bar{2}$	$3 + \bar{2}$	1	7
/ $\bar{4}$	$1 + \bar{2} + \bar{4}$	2	14
/4 $\overline{28}$	$\bar{4}$	/4	28.

In this manner the work proceeds. In dividing 2 by 5, 9, 11, 17, 23, 29 and a few of the larger integers, the $\bar{3}$-sequence is used, i.e. the sequence of fractions $\bar{3}$, $\overline{3}$, $\overline{6}$, $\overline{12}$, ...; but the division by 7 and 13 employs only the $\bar{2}$-sequence $(\bar{2}, \bar{4}, \bar{8}, \ldots)$. It turns out that only in these two cases the $\bar{2}$-sequence produces a simpler result than the $\bar{3}$-sequence. For instance, the use of the $\bar{2}$-sequence would, in calculating $2 : 11$, lead to the result $2 : 11 = \bar{8} + \overline{22} + \overline{88}$, while the $\bar{3}$-sequence gives $2 : 11 = \bar{6} + \overline{66}$, which, having fewer terms and smaller denominators, is obviously to be preferred.

The calculations which have been reproduced here certainly tell their own story. In the case $2 : 7$, the number 4, placed in front of $\overline{28}$, indicates where 28 comes from, viz. from 4×7, the further details being shown in an auxiliary column.

The results of the divisions $2 : n$ are summarized in the following table, which does not include divisors that are divisible by 3, all of which follow the rule $2 : 3k = \overline{2k} + \overline{6k}$.

$2 : 5 = \bar{3} + \overline{15}$	$2 : 53 = \overline{30} + \overline{318} + \overline{795}$
$2 : 7 = \bar{4} + \overline{28}$	$2 : 55 = \overline{30} + \overline{330}$
$2 : 11 = \bar{6} + \overline{66}$	$2 : 59 = \overline{36} + \overline{236} + \overline{531}$
$2 : 13 = \bar{8} + \overline{52} + \overline{104}$	$2 : 61 = \overline{40} + \overline{244} + \overline{488} + \overline{610}$
$2 : 17 = \overline{12} + \overline{51} + \overline{68}$	$2 : 65 = \overline{39} + \overline{195}$
$2 : 19 = \overline{12} + \overline{76} + \overline{114}$	$2 : 67 = \overline{40} + \overline{335} + \overline{536}$
$2 : 23 = \overline{12} + \overline{276}$	$2 : 71 = \overline{40} + \overline{568} + \overline{710}$
$2 : 25 = \overline{15} + \overline{75}$	$2 : 73 = \overline{60} + \overline{219} + \overline{292} + \overline{365}$
$2 : 29 = \overline{24} + \overline{58} + \overline{174} + \overline{232}$	$2 : 77 = \overline{44} + \overline{308}$
	$2 : 79 = \overline{60} + \overline{237} + \overline{316} + \overline{790}$
$2 : 31 = \overline{20} + \overline{124} + \overline{155}$	$2 : 83 = \overline{60} + \overline{332} + \overline{415} + \overline{498}$
$2 : 35 = \overline{30} + \overline{42}$	$2 : 85 = \overline{51} + \overline{255}$
$2 : 37 = \overline{24} + \overline{111} + \overline{296}$	$2 : 89 = \overline{60} + \overline{356} + \overline{534} + \overline{890}$
$2 : 41 = \overline{24} + \overline{246} + \overline{328}$	
$2 : 43 = \overline{42} + \overline{86} + \overline{129} + \overline{301}$	$2 : 91 = \overline{70} + \overline{130}$
$2 : 47 = \overline{30} + \overline{141} + \overline{470}$	$2 : 95 = \overline{60} + \overline{380} + \overline{570}$
$2 : 49 = \overline{28} + \overline{196}$	$2 : 97 = \overline{56} + \overline{679} + \overline{776}$
$2 : 51 = \overline{34} + \overline{102}$	
	$2 : 101 = \overline{101} + \overline{202} + \overline{303} + \overline{606}$

Beginning with $2 : 31$, the form of presentation changes; the calculations are given in abbreviated form. But, what is more important, the method of calculation changes; another idea is introduced. While up to this point, all divisions were carried out by means of the $\bar{2}$-sequence and the $\bar{3}$-sequence, the divisions $2 : 31$ and $2 : 35$ proceeded quite differently, as is seen from the following examples:

What part is 2 of 31? $\overline{20}$ is $1 + \bar{2} + \overline{20}$, $\overline{124}$ is $\bar{4}$, $\overline{155}$ is $\bar{5}$.

Computation:

1	31
$\overline{20}$	$1 + \bar{2} + \overline{20}$
/4 $\overline{124}$	$\bar{4}$
/5 $\overline{155}$	$\bar{5}$

What part is 2 of 35? $\overline{30}$ is $1 + \bar{6}$, $\overline{42}$ is $\bar{3} + \bar{6}$

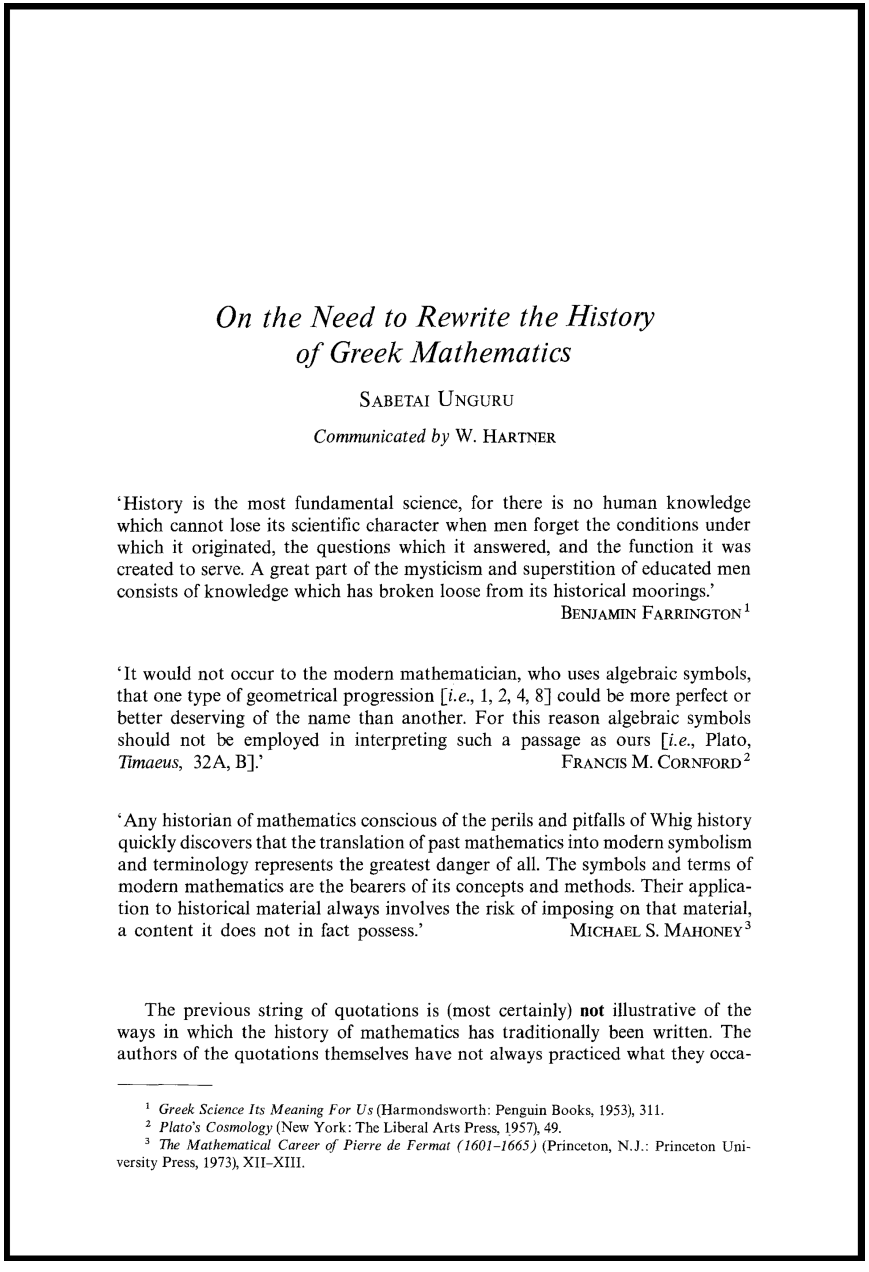
6 7 5

Computation:

1	35
$\overline{30}$	$1 + \bar{6}$
$\overline{42}$	$\bar{3} + \bar{6}$

The start of the computation of $2 : 31$ is easy to account for, since division of 31 by 10, and halving of the result shows that $^1/_{20}$ of 31 is $1 + \bar{2} + \overline{20}$. This fraction is to be increased so as to produce 2. How did the calculator hit upon the idea that this requires $\bar{4} + \bar{5}$? It checks; for the leather scroll has the relation

Rewriting the history of mathematics



On the Need to Rewrite the History of Greek Mathematics

SABETAI UNGURU

Communicated by W. HARTNER

'History is the most fundamental science, for there is no human knowledge which cannot lose its scientific character when men forget the conditions under which it originated, the questions which it answered, and the function it was created to serve. A great part of the mysticism and superstition of educated men consists of knowledge which has broken loose from its historical moorings.'
BENJAMIN FARRINGTON [1]

'It would not occur to the modern mathematician, who uses algebraic symbols, that one type of geometrical progression [*i.e.*, 1, 2, 4, 8] could be more perfect or better deserving of the name than another. For this reason algebraic symbols should not be employed in interpreting such a passage as ours [*i.e.*, Plato, *Timaeus*, 32A, B].'
FRANCIS M. CORNFORD [2]

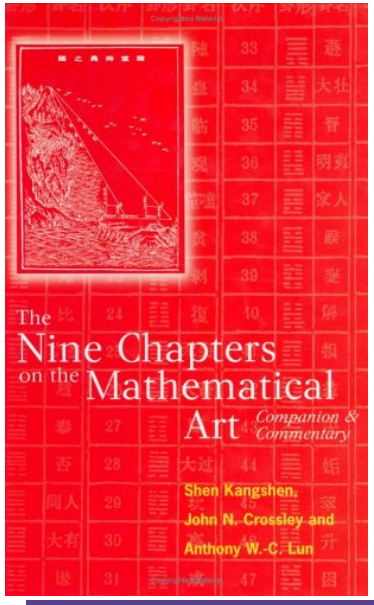
'Any historian of mathematics conscious of the perils and pitfalls of Whig history quickly discovers that the translation of past mathematics into modern symbolism and terminology represents the greatest danger of all. The symbols and terms of modern mathematics are the bearers of its concepts and methods. Their application to historical material always involves the risk of imposing on that material, a content it does not in fact possess.'
MICHAEL S. MAHONEY [3]

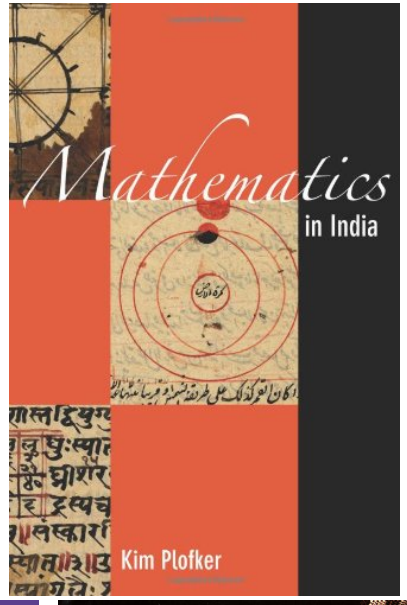
The previous string of quotations is (most certainly) **not** illustrative of the ways in which the history of mathematics has traditionally been written. The authors of the quotations themselves have not always practiced what they occa-

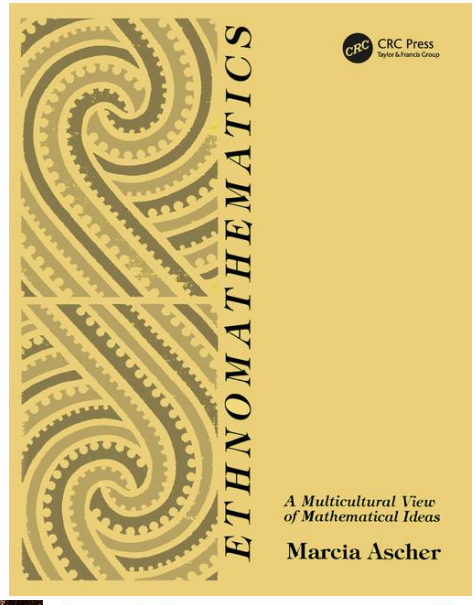
[1] *Greek Science Its Meaning For Us* (Harmondsworth: Penguin Books, 1953), 311.
[2] *Plato's Cosmology* (New York: The Liberal Arts Press, 1957), 49.
[3] *The Mathematical Career of Pierre de Fermat (1601–1665)* (Princeton, N.J.: Princeton University Press, 1973), XII–XIII.

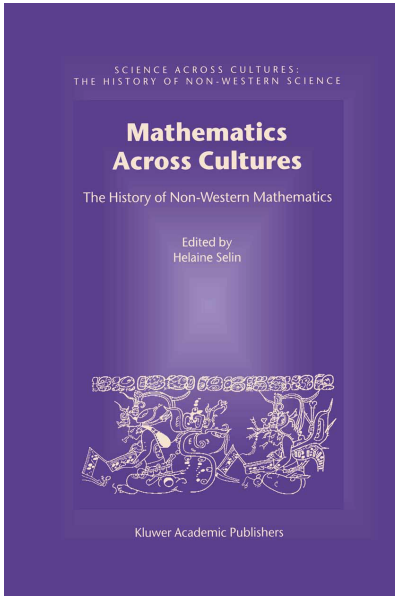
Sabetai Unguru, 'On the need to rewrite the history of Greek mathematics', *Archive for History of Exact Sciences* 15 (1975), 67–114

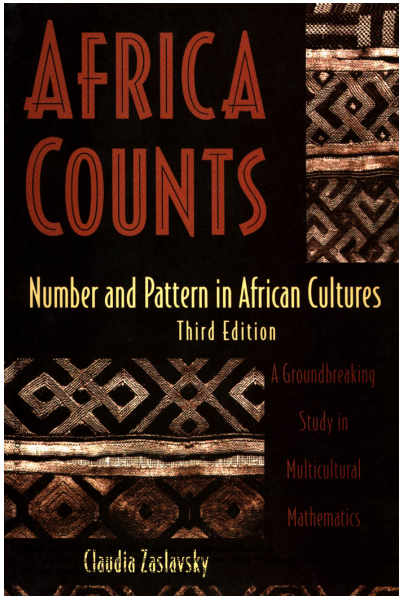
A broader perspective

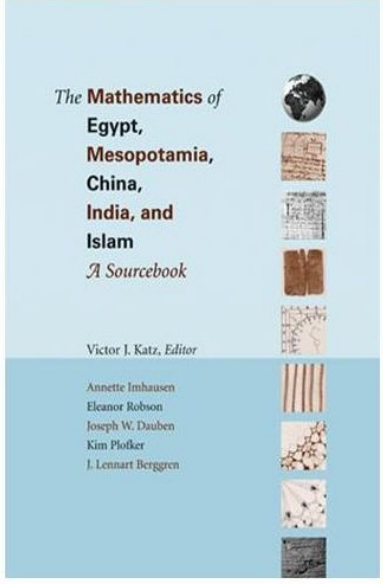








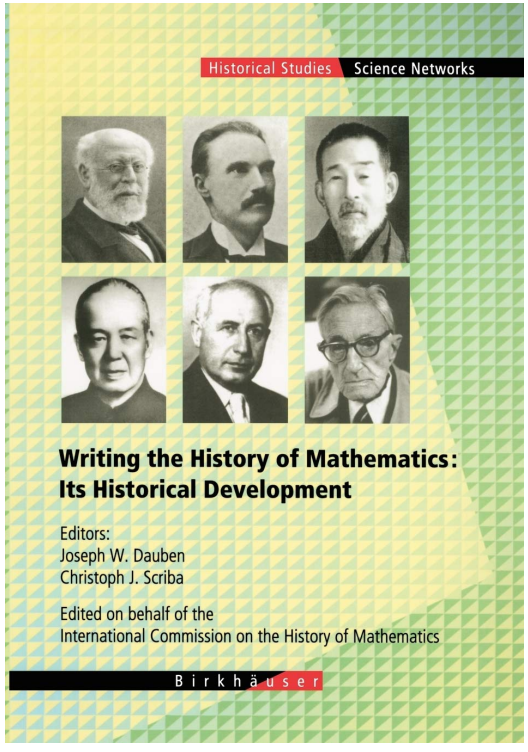


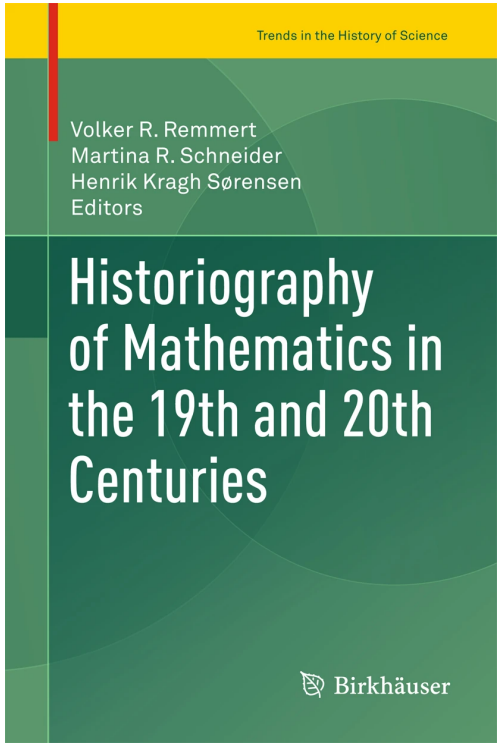


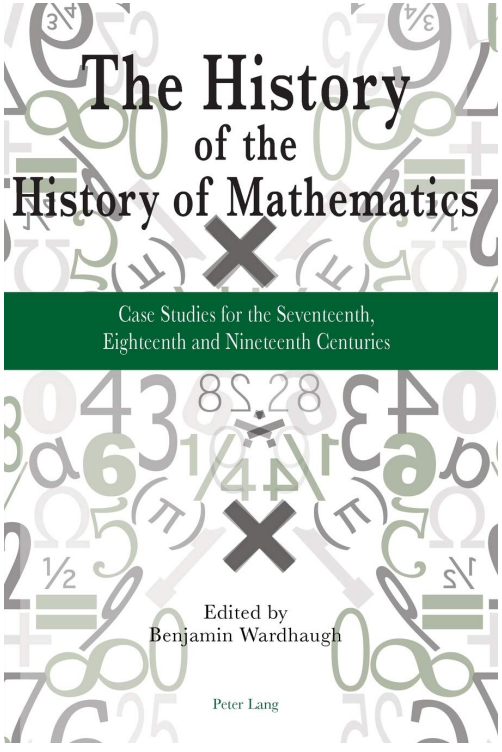
A broader perspective

Brigitte Stenhouse, *Mary Somerville: Being and Becoming a Mathematician*, PhD thesis, The Open University, 2021

Historiography of mathematics: references







How do we know what we know about ancient Egyptian mathematics?

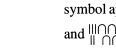
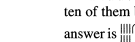
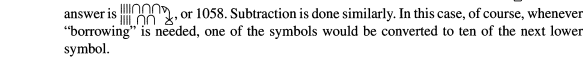
numbers developed to the point where the same digit 1 represented 60 as well. We do not know why the Babylonians decided to have one large unit represent 60 small units and then adapt this method for their numeration system. One conjecture is that 60 is evenly divisible by many small integers. Therefore, fractional values of the "large" unit could easily be expressed as integral values of the "small" unit. The Babylonian base 60 place-value system is still in use in our units for angle and time measurement, units preserved over the centuries in astronomical contexts and today an irreplaceable part of world culture.

There is no record of the written number system of ancient India, but there is literary evidence that numerical symbols did exist. It is only from about the third century B.C.E. that examples of written numbers are available. Originally, the system was mixed. There was a ciphered system similar to the hieratic with separate symbols for the numbers 1 through 9 and 10 through 90. For larger numbers, the system was a multiplicative one similar to the Chinese. For example, the symbol for 200 was a combination of the symbol for 2 and that for 100, and the symbol for 70,000 combined the symbols for 70 and 1000. As will be discussed in Chapter 6, it was in or near India that the modern base 10 place-value system developed, but not until about the seventh century C.E.

1.3 ARITHMETIC COMPUTATIONS

Once their system of writing numbers came into existence, all of the civilizations under discussion devised rules for the basic arithmetic operations—addition, subtraction, multiplication, and division—and as a consequence of the last operation, rules for writing and operating with fractions. These rules may be considered as some of the earliest algorithms.

An **algorithm** is an ordered list of instructions designed to produce an answer to a given type of problem. Ancient peoples produced algorithms of all sorts to handle many different problems. In fact, ancient mathematics can be characterized as algorithmic in nature, as opposed to the Greek mathematics, which emphasized theory. In most of the available documents of ancient mathematics, the author describes a problem to be solved and then proceeds to use an algorithm, either explicit or implicit, to obtain the solution. There is little concern in the documents as to how the algorithm was discovered, why it works, or what its limitations are. Instead, we simply are shown many examples of the use of the algorithm, often in increasingly complex situations. Nevertheless, in our discussion of these algorithms, we will describe the possible origins and justifications of each one and will present the possible answers that the Babylonian, Chinese, or Egyptian scribes gave to their students who asked the eternal question "why?"

In the Egyptian hieroglyphic grouping system, addition is simple enough: Combine the units, then the tens, then the hundreds, and so on. Whenever a group of ten of one type of symbol appears, replace it by one of the next. Hence, to add 783 and 275, put  and  together to get  . Since there are fifteen ∩s, replace ten of them by one 𓆐. This then gives ten of the latter. Replace these by one 𓍢. The final answer is , or 1058. Subtraction is done similarly. In this case, of course, whenever "borrowing" is needed, one of the symbols would be converted to ten of the next lower symbol.

Such a simple algorithm for addition and subtraction is not possible in the hieratic system. For these operations, the mathematical papyri do not provide much evidence; the answers to addition and subtraction problems are merely written down. Most probably, the scribes had addition tables. At some point these would have existed in written form, but a competent scribe would, of course, have memorized them. The scribes presumably used the addition tables in reverse for subtraction problems.

The Egyptian algorithm for multiplication was based on a continual doubling process. To multiply two numbers a and b, the scribe would first write down the pair 1, b. He would then double each number in the pair repeatedly, until the next doubling would cause the first element of the pair to exceed a. Then, having determined the powers of 2 that add to a, the scribe would add the corresponding multiples of b to get the answer. For example, to multiply 12 by 13 the scribe would set down the following lines:

1	12
2	24
4	48
8	96

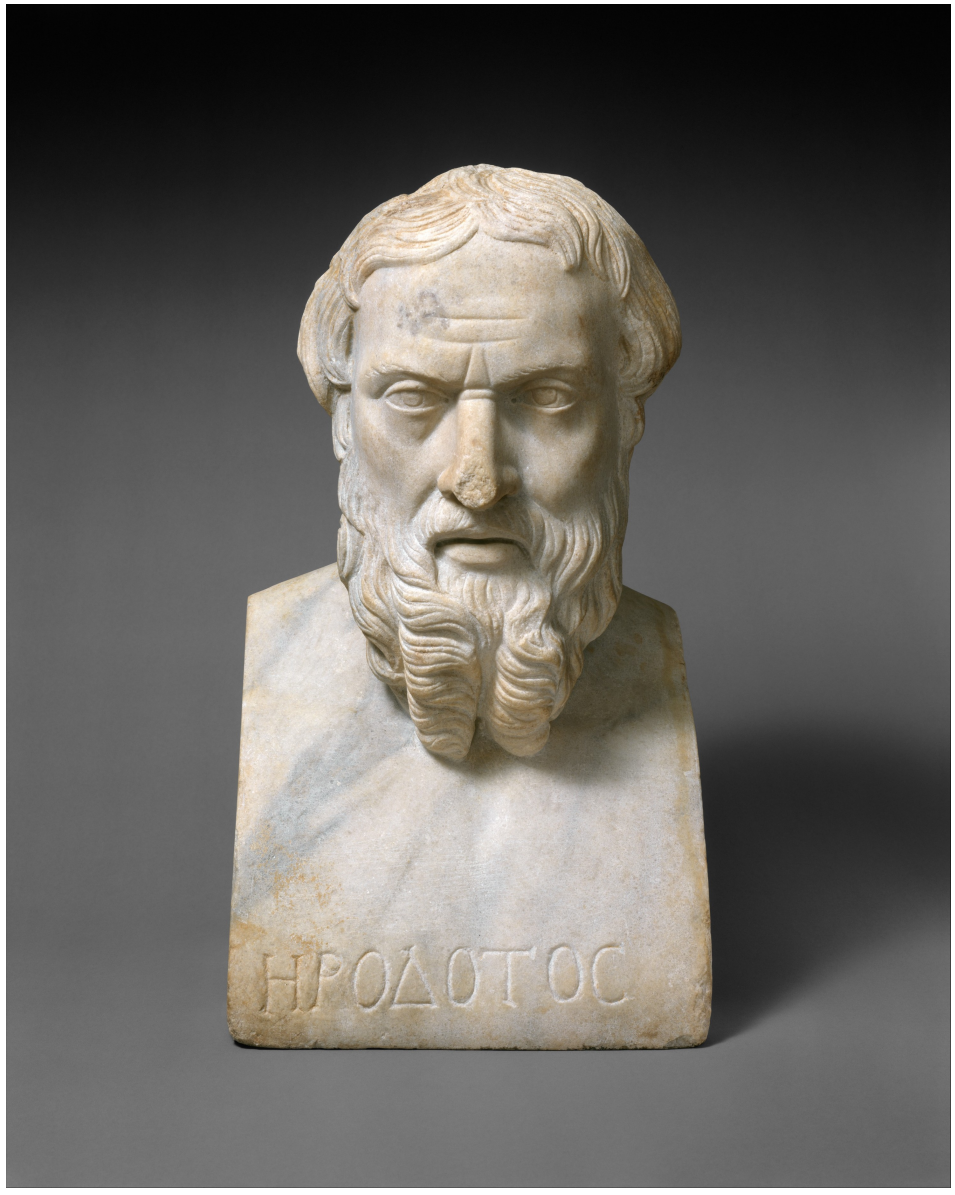
At this point, he would notice that the next doubling would produce 16 in the first column, which is larger than 13. He would then check off those multipliers that added to 13, namely 1, 4, and 8, and add the corresponding numbers in the other column. The result would be written as: Totals 13 156.

As before, there is no record of how the scribe did the doubling. The answers are simply written down. Perhaps the scribe had memorized an extensive 2 times table. In fact, there is some evidence that doubling was a standard method of computation in areas of Africa to the south of Egypt, so it is likely that the Egyptian scribes learned from their southern colleagues.[6] In addition, the scribes were somehow aware that every positive integer could be uniquely expressed as the sum of powers of 2. That fact provides the justification for the procedure. How was it discovered? Our best guess is that it was discovered by experimentation and then passed down as tradition.

Because division is the inverse of multiplication, a problem such as $156 \div 12$ would be stated as "multiply 12 so as to get 156." The scribe would then write down the same lines listed before. This time, however, he would check off the lines having the numbers in the right-hand column that sum to 156; in this case, 12, 48, and 96. Then the sum of the corresponding numbers on the left, namely 1, 4, and 8, would give the answer 13. Of course, division does not always "come out even." When it did not, the Egyptians used fractions.

The kind of fractions that the Egyptians used were unit fractions, or "parts" (fractions with numerator 1), with the single exception of $2/3$, perhaps because these fractions are the most "natural." The fraction $1/n$ (the nth part) is represented in hieroglyphics by the symbol for the integer n with the symbol 𓂝 above. In the hieratic a dot is used instead. Thus $1/7$ is denoted in hieroglyphics by and in the hieratic by . The single exception, $2/3$, had a special symbol: in hieroglyphics and in hieratic. (The former symbol is indicative of the reciprocal of 1 $1/2$.) In the remainder of this text, however, the notation $\overline{n}$ will be used to represent $1/n$ and $\overline{\overline{3}}$ to represent $2/3$.

Herodotus (5th century BC)



Herodotus on Egyptian geometry

It was this king, moreover, who divided the land into lots and gave everyone a square piece of equal size, from the produce of which he exacted an annual tax. Any man whose holding was damaged by the encroachment of the river would go and declare his loss before the king, who would send inspectors to measure the extent of the loss, in order that he might pay in future a fair proportion of the tax at which his property had been assessed. I think this was the way in which geometry was invented, and passed afterwards into Greece ...

Mediaeval Islamic Egypt

Montucla on ancient Egyptian geometry

HISTOIRE
DES
MATHEMATIQUES,

DANS laquelle on rend compte de leurs progrès depuis leur origine jusqu'à nos jours; où l'on expose le tableau & le développement des principales découvertes, les contestations qu'elles ont fait naître, & les principaux traits de la vie des Mathématiciens les plus célebres.

Par M. MONTUCLA, de l'Académie Royale des Sciences & Belles-Lettres de Prusse.

Multi pertransibunt & augebitur scientia. *Bacon.*

TOME PREMIER.

A PARIS,

Chez CH. ANT. JOMBERT, Imprimeur-Libraire du Roi pour l'Artillerie & le Génie, rue Dauphine, à l'Image Notre-Dame.

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assigner à chacun une portion de terre égale à celle qu'il possédoit avant l'inondation. Telle fut, dit-on, l'origine de l'arpentage, premiere ébauche de la Géometrie, à laquelle néanmoins elle a donné le nom: car Géometrie, signifie en Grec, *mesure de la terre*, ou *des terrains*. Je remarque en passant que c'est assez gratuitement qu'on suppose que le Nil confondoit ainsi les limites des possessions; il n'étoit pas bien difficile de lui en opposer d'assez stables ou d'assez profondes pour subsister malgré l'inondation. On ne sçauroit se persuader que l'Egypte fût chaque année ravagée par les eaux: cela s'accorderoit mal avec l'idée d'un pays délicieux, comme celle que nous en donne l'antiquité.

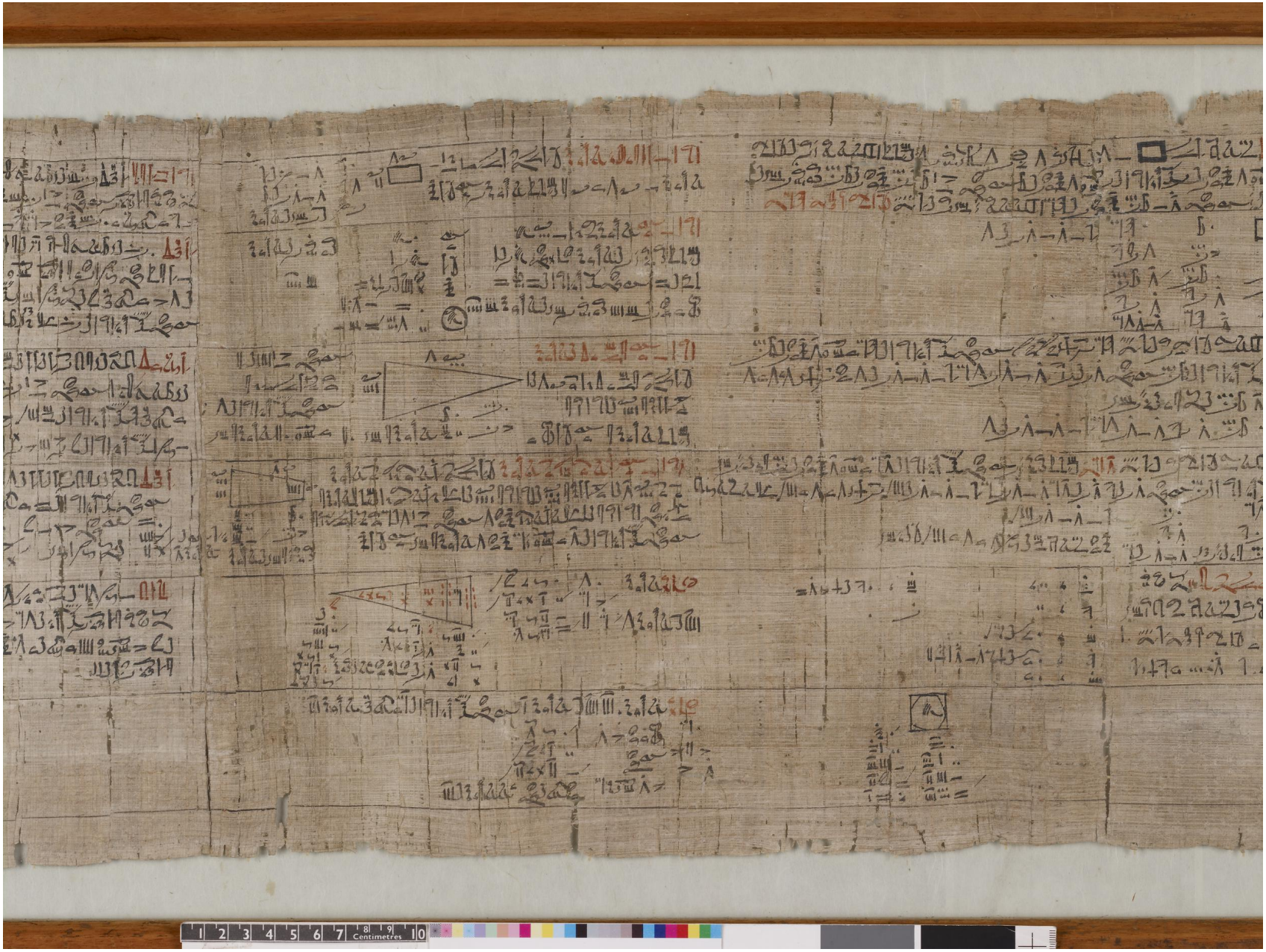
Quelques Ecrivains, parmi lesquels est *Hérodote*, fixent la naissance de la Géometrie au tems où Sesostris (*k*) coupa l'Egypte par des canaux nombreux, & en fit une sorte de répartition générale entre ses habitans. M. *Newton* (*l*) en adoptant le sentiment d'*Hérodote*, dit que ce partage fut fait par le conseil de *Thot*, le Ministre de *Sesostris*, qui est suivant lui *Osiris*. Cette conjecture sur l'emploi & la nature de ce personnage célébre, n'est pas destituée d'autorités anciennes, & s'accorde parfaitement avec l'opinion dont on a parlé ailleurs, que *Theut* étoit l'inventeur des nombres, du calcul & de la Géometrie. En effet, on peut dire que le partage projetté par *Sesostris* exigeant des connoissances Géometriques, son Ministre en jetta à cette occasion les fondemens. Ceci s'accorde encore avec le sentiment qui attribue ces inventions à *Hermes*, autrement le fameux *Mercure Trismegiste*; car tous ces hommes sont probablement les mêmes. Un Ecrivain (*m*) raconte que ce Mercure grava les principes de la Géometrie sur des colonnes qui furent déposées dans de vastes souterrains, & le fabuleux *Jamblique* (*n*) dit que *Pythagore* profita beaucoup de la vûe de ces monumens. Un Auteur enfin cité par *Diogene Laerce*, (*o*) dit que *Mœris*, apparemment ce Prince qui fit creuser le fameux lac de ce nom, pour servir de décharge au Nil, avoit inventé les principes de la Géometrie. On voit facilement le motif de sa conjecture.

(*k*) Herod. *l.* II. XXII.
(*l*) *Chron. ed ann.* 964.
(*m*) Ammian. Marcell. *rerum gest. l.*
(*n*) *In vita Pythagor*, c. 29.
(*o*) *In Pythag.*

Augustus De Morgan (1838)

There is a *stock history* of the rise of geometry . . . that the Egyptians, having their landmarks yearly destroyed by the rise of the Nile, were obliged to invent an art of land-surveying in order to preserve the memory of the bounds of property; out of which art geometry arose. . . . There is no proof whatever . . .

Rhind Mathematical Papyrus



Reconstructing ancient Egyptian mathematics

- Is it valid to use modern mathematical ideas to reconstruct ancient mathematics?
- What do you do if the mathematical and linguistic evidence point in different directions?
- What story were people trying to tell?
- Where does ancient Egyptian mathematics sit within wider stories?

HT reading course: content

Newton's *Universal arithmetick*

The reading course will consist of three parts:

1. the detailed reading and analysis of Newton's methods;
2. the examination of commentaries on Newton's work by British authors Colin Maclaurin and Nicholas Saunderson;
3. the investigation of how Newton's ideas were taken up by continental writers, in particular Leonhard Euler and Joseph-Louis Lagrange.

As during the lecture course, the emphasis will be on the use of original sources. You will therefore read selections from the works of the above authors — plus other relevant materials that may arise.

HT reading course: arrangements

Seminars: weekly classes of an hour and a half each

(Note that these will timetabled with the lectures as two sessions per week, provisionally on Friday mornings, but you only need to attend one of these — sign up as you would for intercollegiate classes.)

Essays: up to 2,000 words to be submitted in advance for discussion in the seminars in weeks 3, 5 and 7

(Further details will appear online during the vacation.)

Assessment: extended essay (3,000 words), details of which will be announced on Monday of week 7. To be submitted by 12 noon on Monday of week 10.

HT reading course: vacation work

Vacation reading for discussion in first-week seminar is now given on the course webpage.

BSHM

The British Society for the History of Mathematics:

www.bshm.ac.uk

BSHM undergraduate essay prize

http://www.bshm.ac.uk/undergraduate-essay-prize

See you next term...